Assignment 5 — Solutions [$Revision: 1.1$]

Question 1

See Figs. 1 and 2.

Question 2

The diffusion equation for the radiative luminosity is

$$L_{r,\mathrm{rad}} = -\frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{1}$$

If this is written in the form

$$L_{r,\mathrm{rad}} = -\frac{4\pi r^2 c}{\kappa\rho}\frac{\mathrm{d}(aT^4/3)}{\mathrm{d}r}, \tag{2}$$

the term in brackets can be recognized as the radiation pressure P_{rad}. Rearranging, the radiation pressure gradient is therefore given by

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\kappa\rho L_{r,\mathrm{rad}}}{4\pi r^2 c}. \tag{3}$$

Now writing the equation of hydrostatic equilibrium in the form

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} + \frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\rho g, \tag{4}$$

the radiation pressure gradient can be eliminated to give the gas pressure gradient as

$$\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} = -\rho g + \frac{\kappa\rho L_{r,\mathrm{rad}}}{4\pi r^2 c}. \tag{5}$$

With a little rearranging, this can be written in the compact form

$$\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} = -\rho g\left[1 - \frac{\kappa L_{r,\mathrm{rad}}}{4\pi g r^2 c}\right]. \tag{6}$$

Based on this expression, it can be seen that there will be a pressure inversion whenever the term in brackets becomes negative; that is, when

$$1 - \frac{\kappa L_{r,\mathrm{rad}}}{4\pi g r^2 c} < 0. \tag{7}$$

Replacing g with GM_r/r^2, this becomes

$$\Gamma \equiv \frac{\kappa L_{r,\mathrm{rad}}}{4\pi GM_r c} > 1, \tag{8}$$

which is the desired result.

Question 3

The radiative temperature gradient is defined by

$$\nabla_{\mathrm{rad}} = \frac{\kappa L_r}{4\pi GM_r c}\frac{3P}{4aT^4} \tag{9}$$

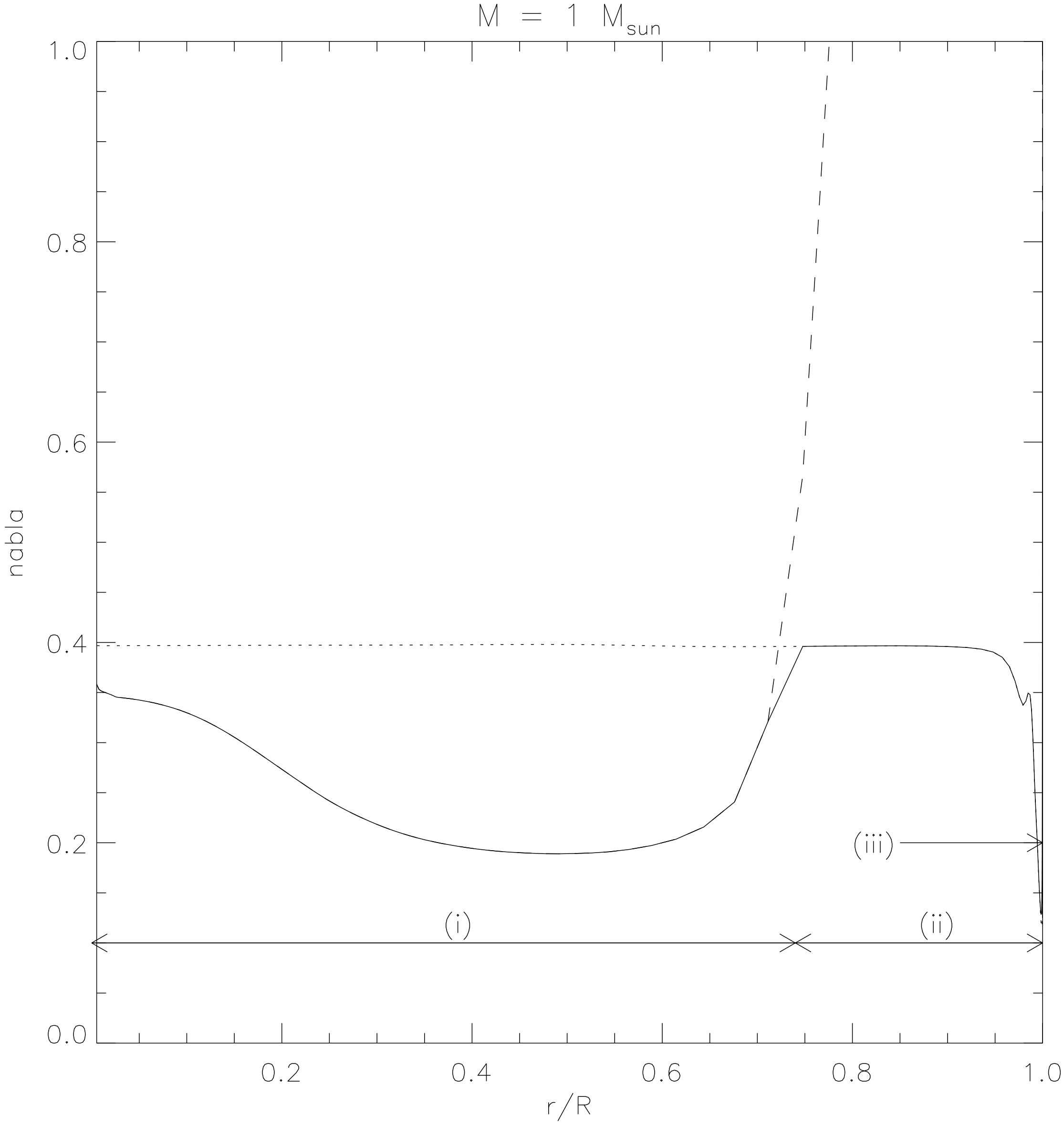

Figure 1: The dimensionless temperature gradients, plotted as a function of fractional radius, for the $1\,M_{\odot}$ model. The solid line is ∇; dashed is $\nabla_{\rm rad}$; and dotted is $\nabla_{\rm ad}$. The arrows and labels indicate the regions where energy is (i) transported by radiation alone, (ii) transported by radiation and efficient convection, or (iii) transported by radiation and inefficient convection.

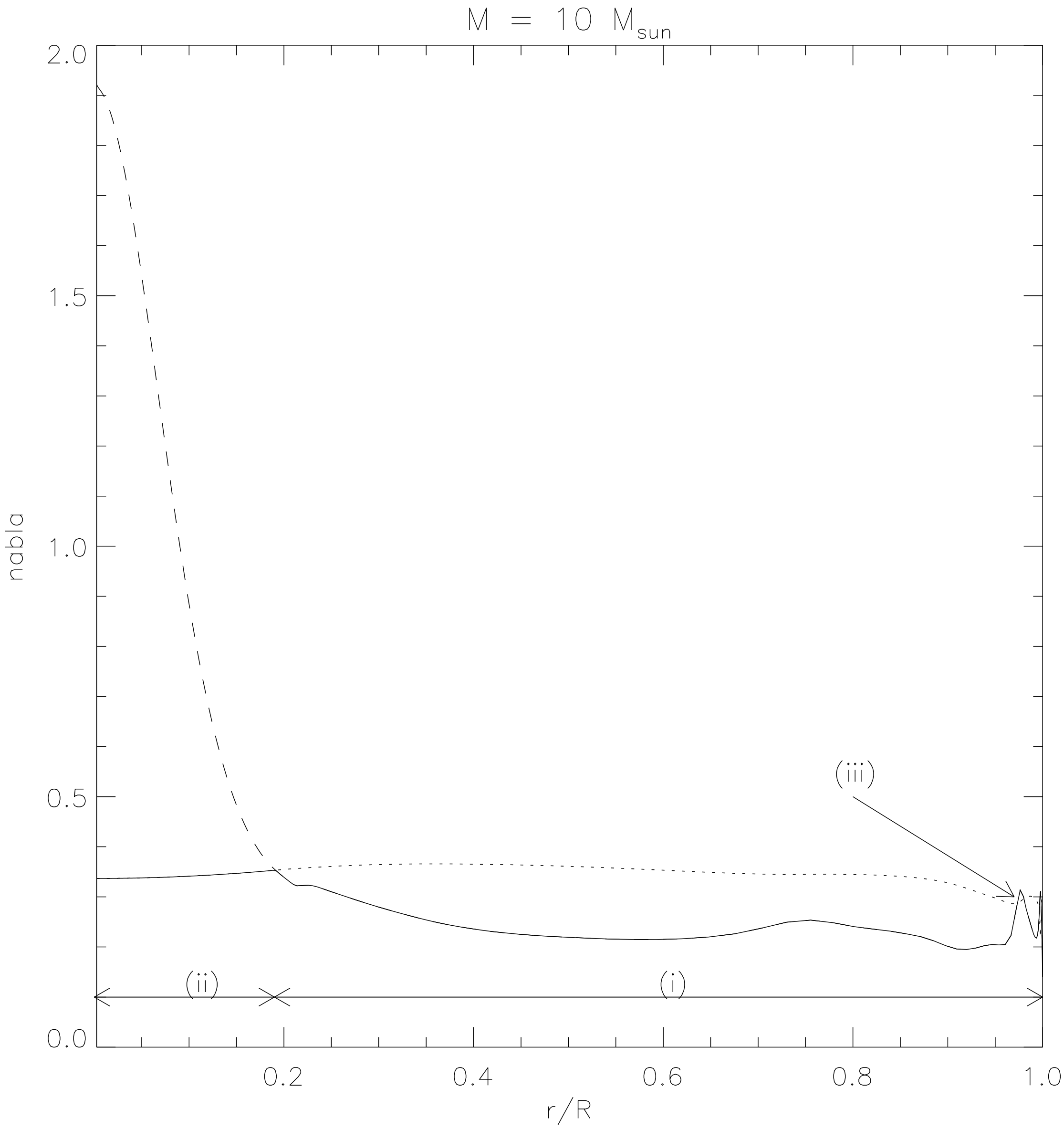

Figure 2: As for Fig. 1, but for the $10\,M_\odot$ model.

(note that the luminosity here is the *total* luminosity, not just the radiative luminosity). With some substitutions, this becomes

$$\nabla_{\rm rad} = \Gamma \frac{L_r}{L_{r,\rm rad}} \frac{P}{4P_{\rm rad}} = \Gamma \frac{L_r}{L_{r,\rm rad}} \frac{1}{4(1-\beta)}. \tag{10}$$

Rearranging,

$$\Gamma = 4\nabla_{\rm rad}(1-\beta)\frac{L_{r,\rm rad}}{L_r}. \tag{11}$$

Question 4

From *Stellar Interiors* (eqns. 3.99 and 3.111),

$$\nabla_{\rm ad} \equiv \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{32 - 24\beta - 3\beta^2}{2(4-3\beta)}. \tag{12}$$

Question 5

Equation (11) indicates that reducing β pushes the star closer to the Eddington limit $\Gamma = 1$. Assuming for the moment that all the luminosity is transported by radiation (i.e., $L_{\rm r} = L_{r,\rm rad}$), then the limit will eventually be reached when

$$\nabla_{\rm rad} = \frac{4}{1-\beta}. \tag{13}$$

From this, we can see that for any $\beta > 0$[1], $\nabla_{\rm rad} > 4$ at the Eddington limit.

Looking now at eqn (12), it is easy to show that for any $\beta > 0$, $\nabla_{\rm ad} < 4$. Accordingly, for any $\beta > 0$, it must hold that

$$\nabla_{\rm rad} > \nabla_{\rm ad} \tag{14}$$

at the Eddington limit. But this can be recognized as the criterion for convection to occur. Hence, we must conclude that convection kicks in at some point *before* the star reaches the Eddington limit.

[1]The $\beta = 0$ limit is unphysical, since the gas pressure must be zero and hence the density zero — there is no matter present!