# Assignment 4 — Solutions [Revision : 1.2]

### Question 1

### Stellar Interiors Q3.1

(i). For the first ionization of helium, the appropriate versions of eqn. (3.24) in *Stellar Interiors* are

$$\exp(\mu_0/kT) = \frac{n_0 h^3}{g_0 (2\pi m_0 kT)^{3/2}} \exp(m_0 c^2/kT),$$
(1)

$$\exp(\mu_1/kT) = \frac{n_1 h^3}{g_1 (2\pi m_1 kT)^{3/2}} \exp(m_1 c^2/kT),$$
(2)

and

$$\exp(\mu_{\rm e}/kT) = \frac{n_{\rm e}h^3}{g_{\rm e}(2\pi m_{\rm e}kT)^{3/2}} \exp(m_{\rm e}c^2/kT).$$
(3)

Chemical equilibrium requires that  $\mu_0 = \mu_1 + \mu_e$ ; hence multiplying the second and third equations and dividing by the first,

$$1 = \exp(\mu_1 + \mu_e - \mu_1) = \frac{n_1 n_e}{n_0} \frac{g_0}{g_1 g_e} \frac{h^3}{(2\pi m_e kT)^{3/2}} \exp(\chi_1/kT),$$
(4)

where  $\chi_1 \equiv (m_1 + m_e - m_0)c^2$  is the first ionization potential of helium, and we've made use of the approximation  $m_1 m_e/m_0 \approx m_e$ .

Rearranging, we arrive at the desired expression for the neutral/ionized ratio,

$$\frac{n_{\rm e}n_1}{n_0} = \frac{g_1g_{\rm e}}{g_0} \frac{(2\pi m_{\rm e}kT)^{3/2}}{h^3} \exp(-\chi_1/kT).$$
(5)

A similar procedure can be used to show that

$$\frac{n_{\rm e}n_2}{n_1} = \frac{g_2g_{\rm e}}{g_1} \frac{(2\pi m_{\rm e}kT)^{3/2}}{h^3} \exp(-\chi_2/kT).$$
(6)

(ii). Number conservation requires that

$$n = n_0 + n_1 + n_2, (7)$$

while charge neutrality means that

$$n_1 + 2n_2 = n_e.$$
 (8)

Dividing the first expression by n, we have

$$z_0 + z_1 + z_2 = 1, (9)$$

where  $z_i \equiv n_i/n$ . Likewise, the second expression can be written as

$$n_{\rm e} = n(z_1 + 2z_2). \tag{10}$$

Thus, the pair of Saha equations become

$$\frac{n(z_1 + 2z_2)z_1}{1 - z_1 - z_2} = \frac{g_1 g_e}{g_0} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp(-\chi_1/kT)$$
(11)

and

$$\frac{n(z_1 + 2z_2)z_2}{z_1} = \frac{g_2g_{\rm e}}{g_1} \frac{(2\pi m_{\rm e}kT)^{3/2}}{h^3} \exp(-\chi_2/kT),\tag{12}$$

with only  $z_1$  and  $z_2$  as unknowns.

- (iii). The above Saha equations are non-linear, and therefore require a multi-variate non-linear root finder to solve. I use a globally-convergent Newton-Raphson method, built on IDL's NEWTON routine (see the course website for the source code). One important trick to obtaining stable solutions is to bring the denominators of the left-hand sides over to the right-hand side. For atomic data, I adopt  $g_1g_e/g_0 = 4$ ,  $g_2g_e/g_1 = 1$ ,  $\chi_1 = 24.6 \text{ eV}$  and  $\chi_2 = 54.4 \text{ eV}$ .
- (iv). Fig 1 shows the ionization state plots for  $\rho = 10^{-4} \,\mathrm{g}\,\mathrm{cm}^3$ .
- (v). The half-ionization points are at  $T \approx 32,000 \,\mathrm{K}$  for He<sup>+</sup>, and  $T \approx 81,000 \,\mathrm{K}$  for He<sup>++</sup> (these values calculated using the same IDL code).

### Stellar Interiors Q3.7

(i). The number distribution function for an electron (Fermi) gas is

$$n(p) = \frac{2}{h^3} \frac{1}{\exp\{[-\mu + E(p)]/kT\} + 1}$$
(13)

(for consistency with the way the question is posed, I've neglected the  $mc^2$  rest-mass energy term; this doesn't affect the results). Let us assume non-relativistic dynamics (since we're looking for a correction to Maxwell-Boltzmann thermodynamics), so that the kinetic energy is given by

$$E(p) = \frac{p^2}{2m}.$$
(14)

Then, in the near-classical limit  $(\mu/kT \ll -1)$ , the distribution function can be approximated to first order by

$$n(p) = \frac{2}{h^3} \left[ 1 - K \exp(p^2/2mkT) \right] K \exp(p^2)/kT),$$
(15)

where, as defined in the question,  $K \equiv \exp(\mu/kT)$ . The total number density is found from

$$n = \int_0^\infty 4\pi p^2 n(p) \,\mathrm{d}p \tag{16}$$

(Stellar Interiors, eqn. 3.10); substituting in the above expression for n(p) and chugging through the integral then gives

$$n = \frac{2(2\pi mkT)^{3/2}}{h^3} (1 - 2^{-3/2}K)K.$$
 (17)

Now making use of the approximation

$$K \approx \frac{n_0 h^3}{2(2\pi m k T)^{3/2}}$$
 (18)

suggested in the question (with the error in the denominator exponent fixed!) gives the desired result,

$$n = n_0 \left( 1 - 2^{-3/2} K \right) \tag{19}$$

(note that only the outer K term has been eliminated here).

(ii). The pressure is found from

$$P = \frac{1}{3m} \int_0^\infty 4\pi p^4 n(p) \,\mathrm{d}p$$
 (20)



Figure 1: The ionization state for helium at  $\rho = 10^{-4} \text{ g cm}^{-3}$ , plotted as a function of temperature. In the upper panel, the solid line shows the neutral He fraction  $z_0$ , while the dotted (dashed) line shows the first- (second-) ionized He fraction  $z_1$  ( $z_2$ ). The lower panel shows  $z_e$ , the number of free electrons per He nucleus.

(Stellar Interiors, eqn. 3.13, with v = p/m). Substituting in the above expression for n(p) gives

$$P = \frac{(2\pi mkT)^{5/2}}{\pi h^3 m} (1 - 2^{-5/2}K)K.$$
 (21)

Again using the approximation for K, this becomes

$$P = n_0 k T \left( 1 - 2^{-5/2} K \right), \tag{22}$$

which is the desired result.

## Question 2

### Stellar Interiors Q4.6

The half-ionization temperatures, calculated using the same IDL code as above for Q3.1, are listed in Table 1. These do indeed correspond to the bumps seen in Fig. 4.3 (but I'm not going to take the time to hand-trace the figure).

$ ho \ ({\rm gcm^{-3}})$	$T_{1/2}$ He <sup>+</sup> (K)	$T_{1/2}$ He <sup>++</sup> (K)
$10^{-4}$	32,000	81,000
$10^{-6}$	22,000	54,000
$10^{-8}$	17,000	40,000

Table 1: Half-ionization temperatures for helium at different densities.

### Stellar Interiors Q4.10

(i). Taking the natural log of eqn. (4.44) of Stellar Interiors,

$$4\ln T = \ln\left(1 + \frac{3\tau}{2}\right) + C,\tag{23}$$

where C is a constant. Differentiating with respect to  $\tau$ ,

$$4\frac{\mathrm{d}\ln T}{\mathrm{d}\tau} = \frac{3}{2+3\tau},\tag{24}$$

which leads directly to the desired result

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\tau} = \frac{3}{8+12\tau}.\tag{25}$$

(ii). For constant opacity, eqn. (4.46) gives

$$P(\tau) = \frac{g_{\rm s}}{\kappa} \int_0^{\tau} \mathrm{d}\tau = \frac{g_{\rm s}}{\kappa} \tau.$$
 (26)

Writing the equation of hydrostatic equilibrium (4.45) as

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}P}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\rho g,\tag{27}$$

we have

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = -\rho g \left(\frac{\mathrm{d}\tau}{\mathrm{d}r}\right)^{-1}.$$
(28)

Since

$$d\tau = -\kappa\rho dr \tag{29}$$

(cf. eqn. 4.8), this becomes

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{g}{\kappa} = \frac{P}{\tau},\tag{30}$$

where the right-most equality comes from the expression above for  $P(\tau)$ . Dividing through by P,

$$\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}\tau} \equiv \frac{\mathrm{d}\ln P}{\mathrm{d}\tau} = \frac{1}{\tau},\tag{31}$$

which is the desired result.

(iii). The temperature gradient  $\nabla$  can be written as

$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \frac{\mathrm{d}\ln T}{\mathrm{d}\tau} \left(\frac{\mathrm{d}\ln P}{\mathrm{d}\tau}\right)^{-1} = \frac{3\tau}{8+12\tau}.$$
(32)

At large optical depths, this limits to

$$\sum_{\tau \to \infty} = 1/4. \tag{33}$$

If  $\Gamma_2 < 4/3$ , then  $\nabla_{ad} < 1/4$ . We then have  $\nabla > \nabla_{ad}$ , indicating that convection will occur.

### Stellar Interiors Q4.11

The Planck function  $B_{\nu}(T)$  is

$$B_{\nu}(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}.$$
(34)

Differentiating with respect to temperature, we have

$$\frac{\partial B_{\nu}}{\partial T} = \frac{2h\nu^3}{c^2} \frac{h\nu}{kT^2} \frac{\exp(h\nu/kT)}{\left[\exp(h\nu/kT) - 1\right]^2}$$
(35)

This is the weighting function in the expression (4.22) for the Rosseland mean opacity. It will afford the most weight to photons when it is maximal. Thus, setting the derivative with respect to  $\nu$  to zero, we have

$$\frac{h^2 \nu_{\max}^3 \left[4kT - h\nu_{\max} \coth(h\nu_{\max}/2kT)\right]}{c^2 k^2 T^3 \left[\cosh(h\nu_{\max}/kT) - 1\right]} = 0.$$
(36)

Rearranging,

$$\nu_{\max} = \frac{kT}{h} \frac{2}{\coth(h\nu_{\max}/2kT)}.$$
(37)

This equation has an approximate solution  $\nu_{\text{max}} \approx kT/h$ , because  $\coth(1/2) = 2.16 \approx 2$  and so the second term on the right-hand side is close to unity. QED.