Assignment 2 — Solutions $[Revision : 1.2]$

Question 1

To obtain the desired density-radius relation, we need to derive an expression for the luminosity in terms of the density and radius, and then combine this with the empirical $L \propto R^3$ luminosity-radius relation.

The stellar luminosity is found by integrating the equation of energy conservation, viz

$$L = \int_0^R \frac{\mathrm{d}L_r}{\mathrm{d}r}\,\mathrm{d}r = \int_0^R 4\pi r^2 \rho\epsilon\,\mathrm{d}r. \tag{1}$$

The nuclear energy release rate depends on the density ρ (assumed constant throughout the star) and the temperature T, via

$$\epsilon = \epsilon_0 \rho T^4 \tag{2}$$

(where ϵ_0 is a constant). Thus, to calculate the stellar luminosity, we're going to need an expression for the temperature as a function of radius.

This expression can be found by first solving the equation of hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho. \tag{3}$$

For a constant-density star, the mass coordinate M_r is simply the mass of a uniform sphere with radius r,

$$M_r = \frac{4\pi r^3 \rho}{3}. \tag{4}$$

Hence,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi G r \rho^2}{3}, \tag{5}$$

and, adopting the surface boundary condition $P(R) = 0$, the pressure distribution throughout the star is

$$P(r) = \frac{2\pi G \rho^2}{3}\left(R^2 - r^2\right). \tag{6}$$

Using the ideal gas law, this is readily transformed into a temperature distribution

$$T(r) = \frac{2\pi G \rho \mu u}{3k}\left(R^2 - r^2\right). \tag{7}$$

Combining the above expression with that for ϵ, the luminosity integral (1) can be evaluated as

$$L = \frac{8192\, G^4 \pi^5 R^{11} u^4 \epsilon_0 \mu^4 \rho^6}{280665\, k^4}, \tag{8}$$

or more simply,

$$L \propto R^{11}\rho^6. \tag{9}$$

With the $L \propto R^3$ relation, this becomes

$$R^3 \propto R^{11}\rho^6, \tag{10}$$

and so

$$\rho \propto R^{-4/3}. \tag{11}$$

Hence, the exponent in the radius-density relation is $\chi = -4/3$, and — because this exponent is negative — we find that bigger/brighter stars are less dense than smaller/dimmer stars. Voila!