Assignment 1 — Solutions [*Revision* : 1.4]

Question 1

(i). We start with the condition of hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho. \tag{1}$$

Because the corona contains negligible mass, the gravity g is simply that provided by the Sun,

$$g = \frac{GM_\odot}{r^2}. \tag{2}$$

Moreover, because the corona is isothermal, the density is directly proportional to the pressure,

$$\rho = \frac{P\mu u}{kT}, \tag{3}$$

where the temperature T is presumed constant. With these two expressions, the equation of hydrostatic equilibrium becomes

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_\odot}{r^2}\frac{P\mu u}{kT}. \tag{4}$$

This is integrated to give

$$\ln P = \frac{GM_\odot}{R_\odot}\frac{\mu u}{kT}\frac{R_\odot}{r} + C \tag{5}$$

where C is the constant of integration. Solving for P,

$$P(r) = P_\odot \exp\left[\frac{GM_\odot}{R_\odot}\frac{\mu u}{kT}\left(\frac{R_\odot}{r} - 1\right)\right], \tag{6}$$

where $P_\odot$ is the pressure at the base of the corona.

(ii). Setting $r \to \infty$ in the above expression, the pressure at infinity is found as

$$P(\infty) = P_\odot \exp\left[-\frac{GM_\odot}{R_\odot}\frac{\mu u}{kT}\right], \tag{7}$$

which is the desired relation.

(iii). From the number density and temperature at the base of the corona, we find $P_\odot = 1.38 \times 10^4\,\mathrm{dyne\,cm^{-2}}$. The pressure far from the Sun is therefore $P(\infty) = 9.25 \times 10^{-3}\,\mathrm{dyne\,cm^{-2}}$.

(iv). The far-from-the-Sun pressure ($9.25 \times 10^{-3}\,\mathrm{dyne\,cm^{-2}}$) is much larger than the ambient pressure in the Local Interstellar Cloud ($8.28\times10^{-14}\,\mathrm{dyne\,cm^{-2}}$). This means that pressure balance cannot be achieved between the two. Instead, the corona will continue to expand into the LIC, resulting in a (non-hydrostatic) wind outflow.

(v). For an adiabatic corona, the equation of hydrostatic equilibrium becomes

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_\odot}{r^2}\frac{P^{3/5}}{K^{3/5}} = -\frac{GM_\odot}{r^2}\frac{P^{3/5}P_\odot^{2/5}\mu u}{kT_\odot} \tag{8}$$

where in the right-most expression the constant K has been rewritten in terms of the pressure $P_\odot$ and temperature $T_\odot$ at the coronal base. Solving for the pressure,

$$P^{2/5} = \frac{GM_\odot}{r}\frac{2P_\odot^{2/5}\mu u}{5kT_\odot} + C, \tag{9}$$

or

$$P(r) = P_\odot \left[\frac{2GM_\odot}{5R_\odot} \frac{\mu u}{kT_\odot} \left(\frac{R_\odot}{r} - 1 \right) + 1 \right]^{5/2}. \tag{10}$$

At some large but finite r, the quantity in square brackets eventually drops to zero. A little before this radius is reached, the coronal pressure will be low enough to match the pressure of the Local Interstellar Cloud, indicating that hydrostatic equilibrium — in balance with the LIC — can be achieved, with no wind outflow.

Physically, the difference between the two cases lies in the hidden ingredient of isothermality: an effectively limitless energy supply able to maintain the corona at a (hot) constant temperature. This energy supply keeps the pressure (and pressure gradients) in an isothermal corona high enough to drive a wind outflow; the kinetic energy required to escape from the parent star's gravitational potential well comes ultimately from the thermal energy (heat) added to maintain isothermality.

Contrast this with an adiabatic corona, whose only supply of energy is the thermal energy it starts out with. This thermal energy is typically less than the gravitational binding energy (as can be seen from the fact that the leading term in the square brackets of eqn. 10 exceeds unity), meaning that the corona remains bound to the star — thus, no wind.

Question 2

***Stellar Interiors* Q1.1**

(i). For luminosity class V, a color index $B - V$ of 1.6 corresponds to an international color index $C = 1.1(B - V) - 0.18 = 1.58$ (Allen 1973, §95); this is closest to a spectral type of M5 (*ibid*, §98).

(ii). The distance to the star is calculated from the parallax as $d \equiv 1/\pi = 4\,\mathrm{pc}$. Its distance modulus is $m_V - M_V \equiv 5\log(d/10\,\mathrm{pc}) = -1.99$, whence the absolute visual magnitude is found as $M_V = 11.8$. The bolometric correction for an M5V star is $B.C. = -2.4$ (Allen 1973, §98), giving the bolometric magnitude as $M_{\mathrm{bol}} \equiv M_V + B.C. = 9.39$. The luminosity can be found by using the Sun as a reference: $M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} = -2.5\log(L/L_\odot)$, with $M_{\mathrm{bol},\odot} = 4.75$ (*Stellar Interiors*, Appendix A), gives $L = 1.4 \times 10^{-2}\,L_\odot$.

(iii). The effective temperature for an M5V star is $T_{\mathrm{eff}} = 2,800K$. The radius is found from $L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4$, as $0.50\,R_\odot$.

(iv). The mass can be estimated from the surface gravity, $g = GM/R^2$. Assuming $g \approx 2.8\,cm\,s^{-2}$ (the same as the Sun) then gives $M \approx 0.26\,M_\odot$. This is close to the $M = 0.22\,M_\odot$ quoted by Allen (1973, §98) for an M5V star.

***Stellar Interiors* Q1.6**

(i). The orbital energy of the planet, added to the star as it accretes, is equal to its circular-orbit kinetic energy,

$$\Delta W = \frac{mv^2}{2} = \frac{GMm}{2R}. \tag{11}$$

(as instructed, we're neglecting gravitational and chemical energies).

(ii). If

$$\Omega = -q\frac{GM^2}{R}, \tag{12}$$

then

$$\Delta\Omega = \Omega \left[2\frac{\Delta M}{M} - \frac{\Delta R}{R} \right] = -q\frac{GM^2}{R} \left[2\frac{m}{M} - \frac{\Delta R}{R} \right] \tag{13}$$

(assuming $\Delta q = 0$).

(iii). The virial theorem relates the change in total and gravitational energies, as

$$\Delta W = \frac{3\gamma - 4}{3(\gamma - 1)} \Delta\Omega. \tag{14}$$

Substituting in the expressions for ΔW and $\Delta\Omega$ gives

$$\frac{GMm}{2R} = -q \frac{3\gamma - 4}{3(\gamma - 1)} \frac{GM^2}{R} \left[2\frac{m}{M} - \frac{\Delta R}{R} \right]. \tag{15}$$

Solving for the fractional radius change,

$$\frac{\Delta R}{R} = 2\frac{m}{M} + \frac{m}{2Mq} \frac{3(\gamma - 1)}{3\gamma - 4}. \tag{16}$$

(iv). With $\gamma = 5/3$, $q = 3/2$, $M = M_\odot$ and $m = M_{\rm Jup}$, $\Delta R = 2.5 \times 10^{-3}\, R_\odot$.

***Stellar Interiors* Q1.10**

For a constant-density sphere, the gravitational binding energy is

$$\Omega = -\frac{3GM^2}{5R}, \tag{17}$$

which for the Sun evaluates to -2.3×10^{48} erg. The corresponding 'mass defect' due to gravitational binding is $\Omega/c^2 = -2.5 \times 10^{27}$ g, giving a fractional mass change $\Delta M_\odot / M_\odot = -1.3 \times 10^{-6}$.

Question 3

See Fig. 1 for the plot.

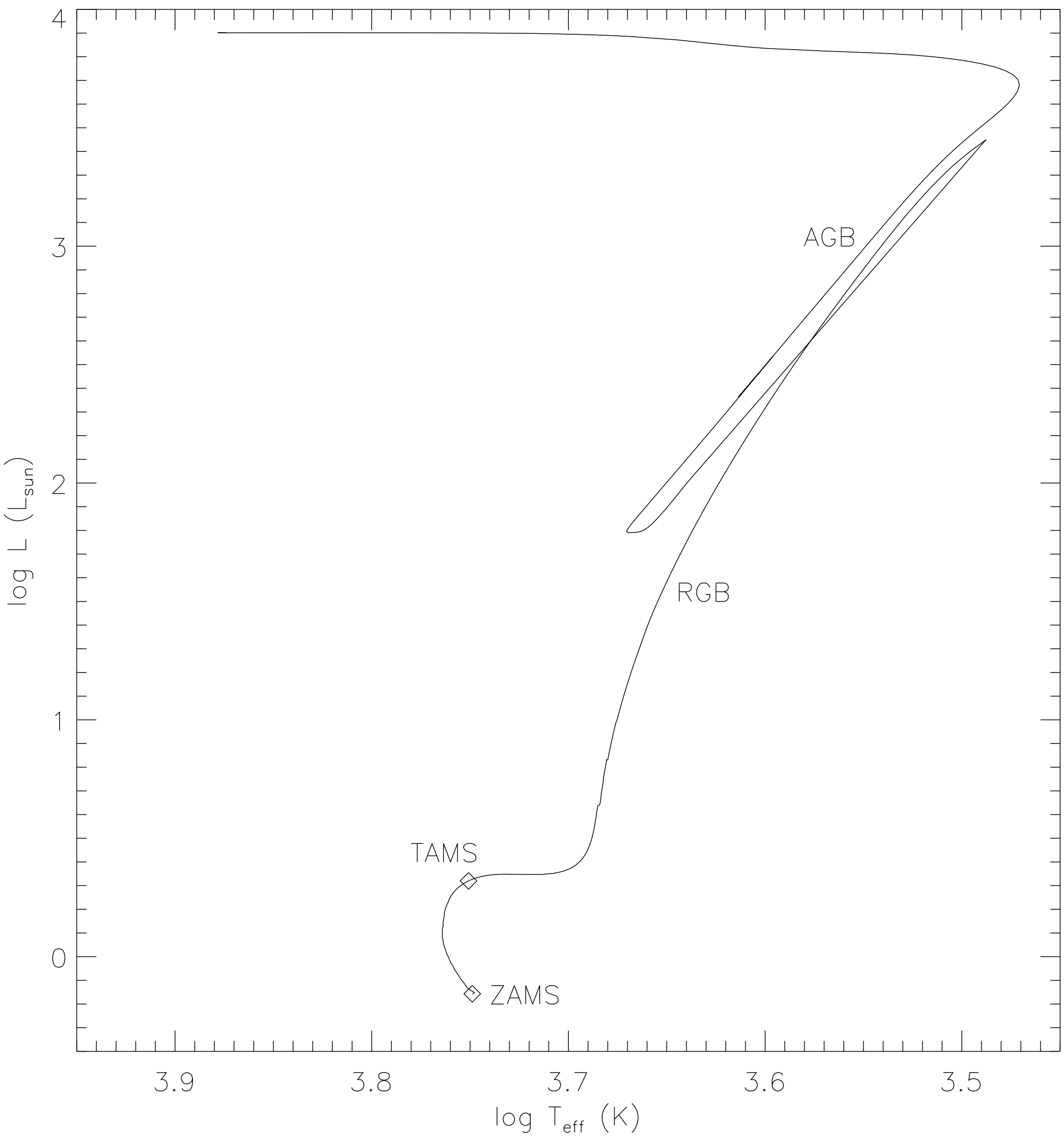

Figure 1: The evolutionary track, in the H-R diagram, of the solar-mass *EZ-Web* model.