Astronomy 715 — Final Exam — Solutions

Question 1

(i). The equation of hydrostatic equilibrium is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho. \tag{1}$$

This corresponds to the scaling

$$\frac{P}{R} \sim \frac{M}{R^2}\rho, \tag{2}$$

where P and rho represent the central pressure and density of the star, respectively, and constant terms (in this case, G) are dropped here and throughout. With $\rho \sim M/R^3$ (from the equation of mass conservation), the scaling becomes

$$P \sim \frac{M^2}{R^4}. \tag{3}$$

Assuming an ideal gas

$$P = \frac{\rho k T}{\mu u}, \tag{4}$$

the corresponding scaling for the temperature can be found as

$$T \sim \frac{P}{\rho}\mu \sim \frac{PR^3}{M}\mu \sim \frac{M}{R}\mu. \tag{5}$$

To determine the luminosity, we now turn to the radiative diffusion equation,

$$L_r = \frac{16\pi r^2 ac T^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{6}$$

This gives the scaling relation

$$L \sim \frac{RT^4}{\kappa\rho} \sim \frac{R^4T^4}{\kappa M}. \tag{7}$$

Eliminating the temperature from this expression,

$$L \sim \frac{R^4M^4}{R^4M}\frac{\mu^4}{\kappa} \sim M^3\frac{\mu^4}{\kappa}, \tag{8}$$

where all dependence on R has dropped out. Thus, because for electron scattering κ is fixed, we have $L \sim M^3\mu^4$, which is the desired result. To derive this, we have not needed the equation of energy conservation, and thus this scaling is independent of the mode of energy generation.

[5 points]

(ii). If the opacity is dominated by bound-free absorption, then it will follow a Kramers' law:

$$\kappa \sim \rho T^{-3.5} \sim \frac{M}{R^3}\frac{R^{3.5}}{M^{3.5}\mu^{3.5}} \sim R^{0.5}M^{-2.5}\mu^{-3.5} \tag{9}$$

When this is substituted into the expression (8) for L, it is clear there will be a residual R term. This term can only be eliminated by introducing further equations — the equation of energy conservation, and the energy generation equation giving ϵ in terms of ρ and T. Thus, the mass-luminosity relation now depends on the mode of energy generation.

[2 points]

(iii). If radiation pressure is dominant, then the ideal-gas equation of state is replaced by

$$P = \frac{aT^4}{3}, \tag{10}$$

so that

$$T^4 \sim P \sim \frac{M^2}{R^4}. \tag{11}$$

Substituting this into the expression (7) for L, we now have

$$L \sim \frac{R^4 M^2}{\kappa R^4} \sim \frac{M}{\kappa}. \tag{12}$$

For fixed κ, this gives us the mass-luminosity relation $L \sim M$, which does not depend on composition. This sort of relation applies to massive main-sequence stars, in which the opacity is primarily electron-scattering and the pressure is dominated by radiation pressure.

[3 points]

Question 2

(i). Plot (a) must be electron-scattering opacity, because κ_λ is independent of wavelength. Plot (b) must be bound-free opacity, because κ_λ shows a prominent edge caused when photons have sufficient energy (or a short enough wavelength) to ionize from a certain bound state. Plot (c) must be bound-bound opacity, because κ_λ is non-zero only at discrete wavelengths corresponding to the energy difference between pairs of bound states.

[3 points]

(ii). The radiative diffusion equation is

$$L_{\rm rad} = -\frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{13}$$

where L_{rmrad} is the radiative flux. The temperature gradient can be written in terms of ∇, as

$$\frac{\mathrm{d}T}{\mathrm{d}r} = T\frac{\mathrm{d}\ln T}{\mathrm{d}r} = T\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\frac{\mathrm{d}\ln P}{\mathrm{d}r} = \frac{T\nabla}{P}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{14}$$

The equation of hydrostatic equilibrium can then be used to eliminate the pressure gradient, so that

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{T\nabla\rho g}{P} \tag{15}$$

Substituting this back into the diffusion equation, and solving for ∇, we have

$$\nabla = \frac{3L_{\rm rad}\kappa P}{16\pi r^2 acT^4 g}, \tag{16}$$

which is the desired result.

[3 points]

(iii). Via the Schwarzschild criterion, convection will commence when $\nabla_{\rm rad} > \nabla_{\rm ad}$ — that is, when the radiative temperature gradient $\nabla_{\rm rad}$ is large. The radiative temperature gradient is defined as the ∇ that would result if *all* the energy were transported by radiative luminosity. Therefore, setting $L_{\rm rad} = L$ in eqn. (16), we have

$$\nabla_{\rm rad} = \frac{3L\kappa P}{16\pi r^2 acT^4 g}. \tag{17}$$

From this expression, we see that $\nabla_{\rm rad}$ will be large (and convection will commence) when (a) κ is large, or when (b) L/r^2 is large.

[2 points]

(iv). Case (a) typically arises in the envelopes of lower-mass main-sequence stars, because the high densities and low temperatures in these envelopes lead to significant bound-free opacity (and hence large κ). Case (b) typically arises in the cores of higher-mass main-sequence stars, because the central concentration of energy generation (a result of CNO-cycle burning) results in a large luminosity near the origin, and hence a large L/r^2.

[2 points]

Question 3

(i). For (a), the hydrogen abundance profile reveals the star is post-main sequence, because $X = 0$ throughout the core. The temperature profile shows that the core is close to isothermal, indicating it must be degenerate. Hence, the star must be low-mass.

For (b) the hydrogen abundance profile reveals that the star is still on the main sequence ($X > 0$ in the core), and also that there is convection occurring in the core. Hence, the star must be high-mass.

[2 points]

(ii). From above, (a) is post-main sequence — specifically, red giant branch, and (b) is main-sequence.

[2 points]

(iii). Because (a) is on the RGB, it must have a low effective temperature and therefore a late spectral type. Because (b) is massive and still on the main sequence, it must have a high effective temperature and therefore an early spectral type.

[2 points]

(iv). (a) has a hydrogen-burning shell at $M_r/M \sim 0.33$. (b) is burning hydrogen in the core, $0 < M_r/M < 0.3$.

[2 points]

(v). (a) will probably become a carbon-oxygen white dwarf; however, if the mass is $\lesssim 0.8\,M_\odot$, it will become a helium white dwarf. If (b) has a mass $\lesssim 9\,M_\odot$, it will become a white dwarf also (CO, or perhaps Ne); above this limit, it will explode as a core-collapse supernova.

[2 points]

Question 4

(i). The Saha equation for hydrogen takes the form

$$\frac{n_e n_{H^+}}{n_{H^0}} = f(T) \tag{18}$$

where $f(T)$ is a monotonic-increasing function of T. At half-ionization, we have

$$\frac{n_e}{2} = f(T_{1/2}), \tag{19}$$

indicating that $T_{1/2}$ must rise (fall) as n_e rises (falls). Looking now at the three cases, the highest $T_{1/2}$ will occur in case (c), because the inclusion of metals (which are generally easier than H to ionize) will introduce lots of free electrons. Likewise, the lowest $T_{1/2}$ will occur in case (b), because the inclusion of helium (which is harder than H to ionize) will cause a deficit of free electrons.

[2 points]

(ii). At zero temperature, $n(p)$ is given by

$$n(p) = \frac{g}{h^3} 4\pi p^2 \tag{20}$$

up to the Fermi momentum $p_{\rm F}$, at which point it drops to zero. See Fig. 1. Note that this question wasn't worded well, and answers that give $n(p) = 1/h^3$ up to $p_{\rm F}$ will also be considered valid.

[2 points]

(iii). At zero temperature (implied in the question), the total number density is given by

$$n = \frac{4\pi g}{h^3} \int_0^{p_{\rm F}} p^2 \, {\rm d}p. \tag{21}$$

Integrating,

$$n = \frac{4\pi g}{3h^3} p_{\rm F}^3. \tag{22}$$

Rearranging,

$$p_{\rm F} = \left(\frac{3h^3 n}{4\pi g} \right)^{1/3}, \tag{23}$$

which is the final result.

[3 points]

(iv). For a Bose-Einstein gas at temperature T,

$$n(p){\rm d}p = \frac{g}{h^3} \frac{4\pi p^2}{\exp([E(p) - \mu]/kT) - 1} {\rm d}p \tag{24}$$

In applying this to photons, we set $\mu = 0$ (since photon numbers are not conserved), and $g = 2$ (to account for the two possible photon polarizations). Hence

$$n(p){\rm d}p = \frac{2}{h^3} \frac{4\pi p^2}{\exp(E(p)/kT) - 1} {\rm d}p \tag{25}$$

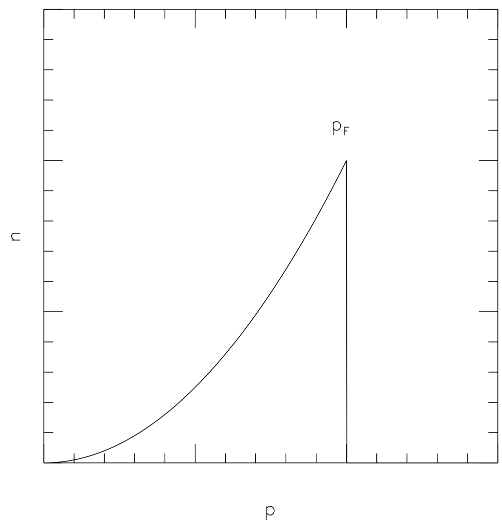

Figure 1: The distribution function for a Fermi gas at zero temperature, showing the Fermi momentum p_{F}. The axis scales are arbitrary.

This is the number of photons per unit volume in a momentum interval dp. The spectral energy density can be calculated by multiplying by the energy per photon, so that

$$u_\nu \mathrm{d}\nu = n(p)E(p)\mathrm{d}p \tag{26}$$

Thus,

$$u_\nu = n(p)E(p)\frac{\mathrm{d}p}{\mathrm{d}\nu} \tag{27}$$

For photons, $E(p) = pc = h\nu$, and also

$$\frac{\mathrm{d}p}{\mathrm{d}\nu} = \frac{h}{c}. \tag{28}$$

Therefore, the spectral energy density is

$$u_\nu = \frac{8\pi h^2\nu^2 h\nu}{h^3c^2}\frac{1}{\exp(h\nu/kT)-1}\frac{h}{c} = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1}. \tag{29}$$

which is the final result.

[3 points]