What is the Virial Theorem?

The *virial theorem* (from the Latin word 'vis', meaning 'force') is a relationship between the total gravitational and thermal energies of stars[1]that are in hydrostatic equilibrium. It is one of the central concepts of stellar astrophysics, as it allows us to understand how stars respond to the loss (or gain) of energy.

[1] There also exist versions of the virial theorem as applied to systems of particles bound by gravity (e.g., individual galaxies in a galaxy cluster); these versions have a similar form to the one we derive here, but the derivation process is rather different.

Derivation of the Virial Theorem

To derive the virial theorem, we start with the equation of hydrostatic equilibrium,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm}{r^2}\rho. \tag{1}$$

Multiply both sides by $4\pi r^3$ and integrate with respect to r over the whole star[2] , to yield

$$\int_0^R \frac{\mathrm{d}P}{\mathrm{d}r} 4\pi r^3 \;\mathrm{d}r = -\int_0^R \frac{Gm}{r} 4\pi r^2 \rho \;\mathrm{d}r \tag{2}$$

[2] A common question here is: why $4\pi r^3$? The answer is that this factor transforms the right-hand side of the hydrostatic equilibrium equation (1) into the integrand appearing in the definition of the gravitational potential energy U.

Using the mass equation (eqn. 2 of *Handout* IV), we transform the right-hand side into an integral with respect to interior mass m,

$$-\int_0^R \frac{Gm}{r} 4\pi r^2 \rho \;\mathrm{d}r = -\int_0^M \frac{Gm}{r} \;\mathrm{d}m \equiv U, \tag{3}$$

where the second equality follows from the definition of the gravitational potential energy (cf. eqn. 7 of *Handout* IV). Likewise, we integrate the left-hand side by parts, and then also transform it into an integral with respect to m,

$$\int_0^R \frac{\mathrm{d}P}{\mathrm{d}r} 4\pi r^3 \;\mathrm{d}r = \left[4\pi r^3 P\right]_0^R - 3\int_0^R 4\pi r^2 P \;\mathrm{d}r = -3\int_0^M \frac{P}{\rho} \;\mathrm{d}m, \tag{4}$$

where the second equality follows from neglecting the surface pressure $P(R)$.

If we now assume the stellar material is described by an ideal-gas equation of state, then from *Handout* VI we have

$$\frac{P}{\rho} = \frac{k_\mathrm{B} T}{\mu \mathrm{m_H}} = (\gamma - 1)u, \tag{5}$$

and so

$$-3\int_0^M \frac{P}{\rho} \;\mathrm{d}m = -3\int_0^M (\gamma - 1)u \;\mathrm{d}m = (3 - 3\langle\gamma\rangle)H, \tag{6}$$

where in the second equality we have taken γ outside the integral sign, replacing it with its mass-weighted average $\langle\gamma\rangle$, and moreover have introduced the total thermal energy of the star as

$$H \equiv \int_0^M u \;\mathrm{d}m. \tag{7}$$

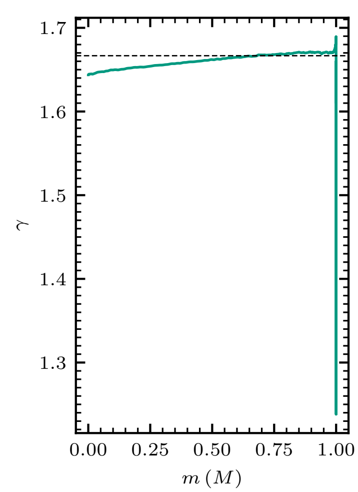

Figure 1: The ratio of specific heats γ, plotted as a function of interior mass m for the *MESA* model of the present-day Sun. For this model, the mass-weighted average of γ is $\langle\gamma\rangle = 1.661$, very close to the fiducial value $\gamma = 5/3 = 1.667$ (shown by the dashed horizontal line).

Putting eqns. (2), (3) and (6) together, we arrive at the final result

$$H = \frac{U}{3 - 3\langle\gamma\rangle}, \tag{8}$$

which is the *virial theorem for stars*. In the common case where $\langle\gamma\rangle = 5/3$ (see, e.g., Fig. 1), it reduces to the even-simpler form

$$H = -\frac{U}{2}. \tag{9}$$

Understanding the Virial Theorem

The virial theorem (8) appears quite straightforward, but don't be deceived by appearance. It's quite unusual for the internal energy of an object to be tied in any way to its gravitational potential energy[3]. However, the force balance between gravity and pressure, for a star *in hydrostatic equilibrium* and following *an ideal gas law*, creates an inescapable link between internal and gravitational potential energies: U and H are not independent quantities, but tied together by eqn. (8). Fig. 2 demonstrates this link in action for the Sun.

[3] Consider, for instance, the objects you see around you as you read this – is there any relationship between their internal and potential energies? Almost certainly not!

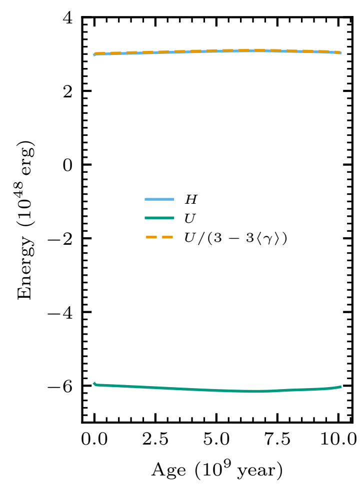

Figure 2: The thermal energy H and gravitational potential energy U, plotted as a function of age for a *MESA* model of the Sun evolving through the main-sequence phase. Also plotted is the quantity $U/(3 - 3\langle\gamma\rangle)$, evaluated using the value $\langle\gamma\rangle = 1.661$ taken from Fig. 1. By the virial theorem (8), this quantity should match H — and clearly, it does.

Virial Theorem and Stellar Energy

The virial theorem leads to a couple of important results when applied to the total stellar energy,

$$E \equiv H + U \tag{10}$$

If we combine this with eqn. (8), we obtain a pair of complementary expressions for E in terms of H alone and U alone:

$$E = (4 - 3\langle\gamma\rangle)H, \qquad E = \frac{4 - 3\langle\gamma\rangle}{3 - 3\langle\gamma\rangle}U. \tag{11}$$

For the star to be stably bound E must be negative[4] . Since H is always positive, the left expression here tells us that $\langle\gamma\rangle > 4/3$ for stability. Fig. 1 confirms that this criterion is satisfied in the present-day Sun.

The two expressions (11) also indicate a counterintuitive property of stars. Assuming that $\langle\gamma\rangle > 4/3$, then a change $\Delta E < 0$ in total energy must be accompanied by a change $\Delta U < 0$ in the gravitational potential energy, but a change $\Delta H > 0$ in the thermal energy. Therefore, as the star loses *total* energy, its *thermal* energy increases — in some sense, it gets hotter. This may seem to violate conservton of energy, but it doesn't.

[4] If instead $E > 0$ then the star has sufficient energy to spontaneously unbind, dispersing the stellar material to infinity.

Further Reading

Kippenhahn, Weigert & Weiss, §3; *Prialnik*, §2.4.