General Equation of State

The pressure P and density ρ of stellar material are connected by the *equation of state* (EOS) governing the material. Most generally, the EOS is written

$$P = P_{\rm ion} + P_{\rm el} + P_{\rm rad} \qquad (1)$$

where $P_{\rm ion}$ is the pressure due to ions[1], $P_{\rm el}$ the pressure due to free electrons resulting from ionization, and $P_{\rm rad}$ the radiation pressure due to photons. All three of these depend on ρ and the temperature T of the material (see Fig. 1).

[1] Here, we use 'ions' generically to denote atoms in any ionization state — whether neutral (when the atoms have their full complement of bound electrons), fully ionized (when the atoms have lost their bound electrons into the pool of free electrons), or some partially ionized state.

Ideal-Gas Equation of State

For low- and intermediate-mass stars on the main sequence (including the Sun), the radiation pressure is negligible, and the ion and free-electron pressures follow the ideal-gas law[2]. Thus, we can write

$$P_{\rm ion} = n_{\rm ion}\, k_{\rm B}\, T, \qquad P_{\rm el} = n_{\rm el}\, k_{\rm B}\, T, \qquad (2)$$

where $n_{\rm ion}$ is the number density of ions, and $n_{\rm el}$ the number density of free electrons. Substituting these expressions into the general EOS (1), and neglecting the $P_{\rm rad}$ term, we obtain

$$P = n\, k_{\rm B}\, T, \qquad (3)$$

where $n \equiv n_{\rm ion} + n_{\rm el}$ is the total particle number density. Often, we further eliminate n to write the EOS as

$$P = \frac{\rho\, k_{\rm B}\, T}{\mu\, {\rm m_H}}, \qquad (4)$$

where we introduce the *mean molecular weight*[3]

$$\mu \equiv \frac{\rho}{n\, {\rm m_H}} \qquad (5)$$

as the average mass per particle (ions *and* free electrons), expressed in units of the hydrogen atomic mass $\rm m_H$.

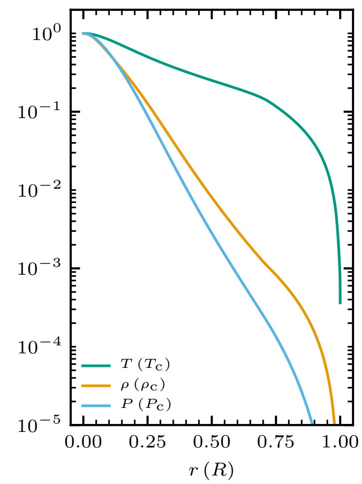

Figure 1: The temperature T, density ρ and pressure P (in units of their central values $T_c = 1.58 \times 10^7\,$K, $\rho_c = 160\,{\rm g\,cm^{-2}}$, $P_c = 2.42 \times 10^{17}\,{\rm dyn\,cm^{-2}}$), plotted as a function of radial coordinate r for a *MESA* model of the present-day Sun.

[2] In an ideal gas, collisions between particles are perfectly elastic, and there are no inter-molecular forces. The ideal-gas law is often written as $PV = N\, k_{\rm B}\, T$, where V is the volume and N is the number of particles contained in the volume. However, when dealing with stellar interiors it is more convenient to use an intrinsic version of the law: $P = n\, k_{\rm B}\, T$, where $n \equiv N/V$ is the number density of particles.

[3] In spite of its name, the mean molecular weight has nothing to do with molecules or weight.

Composition & Mean Molecular Weight

The mean molecular weight of stellar material depends both on its composition and its ionization state. Composition is usually specified by *mass fractions*, which quantify what fraction by mass is composed of a given element. By convention, X denotes the mass fraction of hydrogen, Y the mass fraction of helium, and $Z \equiv 1 - X - Z$ the mass fraction of metals[4].

[4] Another weird nomenclature choice: 'metals', to astronomers, are all elements that aren't hydrogen and helium.

The mean molecular weight in the neutral limit is approximated by

$$\mu \approx \left[X + \frac{Y}{4} + \frac{Z}{12}\right]^{-1} \tag{6}$$

The corresponding value in the fully ionized limit is likewise approximated by

$$\mu \approx \left[2X + \frac{3Y}{4} + \frac{Z}{2}\right]^{-1} \tag{7}$$

A later handout will discuss the origin of these formulae, and how to evaluate μ for partially ionized samples. Fig. 2 shows how these formulae apply to the present-day Sun.

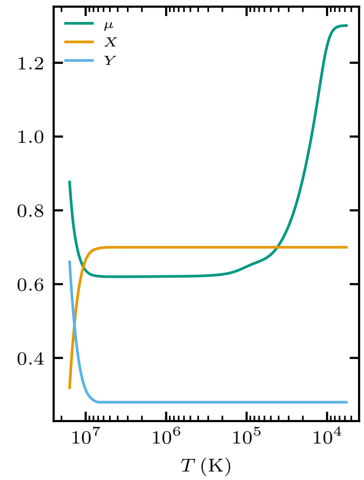

Figure 2: The mean molecular weight μ, and hydrogen (X) and helium (Y) mass fractions, plotted as a function of temperature T for the *MESA* model of the present-day Sun (cf. Fig. 1). The composition of the Sun at birth was $X = 0.70$, $Y = 0.28$ and $Z = 0.02$, and these values still apply for $T \lesssim 8 \times 10^6$ K. At higher temperatures, however, the hydrogen composition is depleted, and the helium composition is enriched; this is the result of the ongoing fusion of hydrogen into helium. With declining temperature below $T \approx 2 \times 10^5$ K, the transition from fully ionized to neutral causes the increase in μ from the value $\mu \approx 0.62$ given by eqn. (7), to the value $\mu \approx 1.30$ given by eqn. (6).

Isothermal & Adiabatic Changes

Sometimes, we wish to know how stellar material responds to changes in its thermodynamic state. For an *isothermal* change the temperature of the material remains constant; then, the pressure follows the relation

$$P = K_{\rm iso}\,\rho, \tag{8}$$

where the constant $K_{\rm iso}$ is set by the temperature and composition of the material. Likewise, for an *adiabatic* change there is no heat absorbed or released by the material; then, the pressure follows the relation

$$P = K_{\rm ad}\,\rho^{\gamma}, \tag{9}$$

where the constant $K_{\rm ad}$ is set by the *initial* temperature and composition of the material. In both the neutral and fully ionized limits, the exponent γ (which is formally known as the 'ratio of specific heats') takes the value 5/3; however, in the partially ionized case its value is usually closer to 1.

Internal Energy

For the ideal-gas EOS (4), the internal (or thermal) energy per unit mass is

$$u = \frac{1}{\gamma - 1}\frac{k_{\rm B}T}{\mu\,{\rm m_H}} \approx \frac{3k_{\rm B}T}{2\mu\,{\rm m_H}}, \tag{10}$$

where the second equality applies when $\gamma = 5/3$.

Further Reading

Kippenhahn, Weigert & Weiss, §§4.1–4.2; *Ostlie & Carroll*, §10.2; *Prialnik*, §§3.1–3.1.