The Momentum Conservation Equation

Consider a vertically oriented cylindrical volume element with cross-sectional area dA, extending from radial coordinate $r_{\rm a}$ out to radial coordinate $r_{\rm b}$ (see Fig. 1). Denoting the net radial force on this element as f, and the net radial momentum of the element as p, Newton's second law[1] applied to the element is

$$f = \frac{\mathrm{d}p}{\mathrm{d}t} \tag{1}$$

The momentum of the element can be evaluated via the integral

$$p = \int_{r_{\rm a}}^{r_{\rm b}} \rho v_r \,\mathrm{d}r\,\mathrm{d}A \tag{2}$$

where v_r is the radial velocity. The force on the element comes from a combination of pressure P acting on the lower and upper ends of the cylinder, and the gravitational acceleration g acting throughout:

$$f = \underbrace{\Big[P(r_{\rm a}) - P(r_{\rm b})\Big]\,\mathrm{d}A}_{\text{pressure force}} + \underbrace{\int_{r_{\rm a}}^{r_{\rm b}} g\rho \,\mathrm{d}r\,\mathrm{d}A}_{\text{gravitational force}}. \tag{3}$$

Applying the fundamental theorem of calculus[2], we can write the pressure term as an integral:

$$f = -\int_{r_{\rm a}}^{r_{\rm b}} \frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}r\,\mathrm{d}A + \int_{r_{\rm a}}^{r_{\rm b}} g\rho\,\mathrm{d}r\,\mathrm{d}A. \tag{4}$$

Substituting this and eqn. (2) back into Newton's second law (1), and dividing through by dA, we arrive at the *momentum conservation equation* for the element:

$$-\int_{r_{\rm a}}^{r_{\rm b}} \left[\frac{\mathrm{d}P}{\mathrm{d}r} - \rho g\right] \mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}t}\int_{r_{\rm a}}^{r_{\rm b}} \rho v_r\,\mathrm{d}r \tag{5}$$

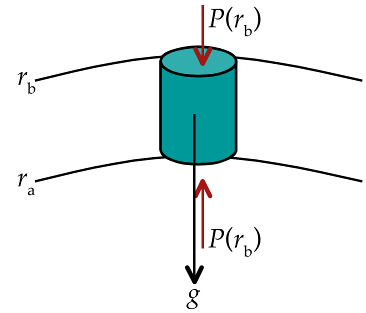

Figure 1: A cylindrical volume element with cross sectional area dA, extending between radial coordinates $r_{\rm a}$ to $r_{\rm b}$. This element is acted on by both pressure (P) and gravitational forces (g), as shown.

[1] While the second law is often given as *force is mass times acceleration*, it is more correctly stated as *force is rate-of-change of momentum*.

[2] This is the theorem that

$$\int_a^b \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y(b) - y(a).$$

Hydrostatic Equilibrium

Now let's suppose that the velocity throughout the star vanishes, so the right-hand side of the momentum conservation equation (5) is zero for every possible choice of $r_{\rm a}$ and $r_{\rm b}$. It then follows that the integrand on the left-hand side of the equation must also be zero; that is,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \rho g \tag{6}$$

This is the equation of *hydrostatic equilibrium*. It establishes the basic condition that must be satisfied at every point in a star, in order for the stellar material to remain in static (zero velocity) equilibrium. In words, *the outward force due to the pressure gradient must balance the inward force due to gravity*. Fig. 2 demonstrates hydrostatic equilibrium in action for a model of the Sun.

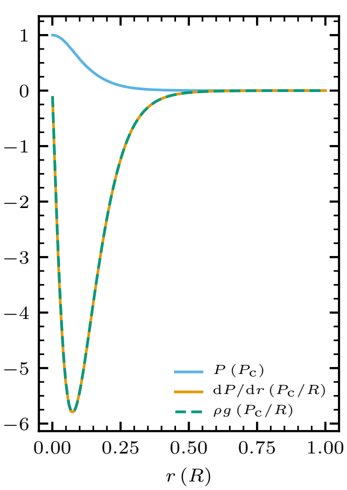

Figure 2: The pressure P (in units of its central value $P_{\rm c} = 2.42 \times 10^{17}$ dyn cm^{-2}), its gradient dP/dr and the gravitational force per unit volume ρg (both in units of $P_{\rm c}/R$), plotted as a function of radial coordinate r for a *MESA* model of the present-day Sun. The close match between dP/dr and ρg indicates that the model is in hydrostatic equilibrium.

Central Pressure

The hydrostatic equilibrium equation (6) provides us with a simple way to place a lower limit on the central pressure P_{c} of a star. Integrating the equation, we can write[3]

$$P_{\mathrm{s}} - P_{\mathrm{c}} = \int_0^R g\rho \, \mathrm{d}r = -\int_0^M \frac{Gm}{4\pi r^4} \, \mathrm{d}m, \tag{7}$$

The surface pressure P_{s} is much smaller than P_{c}, and can be neglected to yield

$$P_{\mathrm{c}} = -\int_0^M \frac{Gm}{4\pi r^4} \, \mathrm{d}m > \frac{GM^2}{8\pi R^4}. \tag{8}$$

Applying this to the present day Sun, we find $P_{\mathrm{c}} > 4.48 \times 10^{14}$ $\mathrm{dyn\,cm^{-3}} = 4.42 \times 10^8$ atm. Comparing this inequality against the actual value of P_{c} (see Fig. 2), we can see that its a pretty loose lower limit.

[3] In the second equality we have used the mass equation (eqn. 2 of *Handout* iv) to switch to an integral over interior mass m, and likewise expressed the gravity in terms of m and r.

The Dynamical Timescale

Imagine that all pressure forces in a star suddenly vanish. Then, the star will begin to collapse under its own gravitational force. The initial acceleration of the surface layers is

$$g_{\mathrm{s}} = \frac{GM}{R^2}$$

Assuming that this acceleration remains constant, the time taken for these surface layers to collapse down to the origin is

$$\Delta t = \sqrt{\frac{2R}{g_{\mathrm{s}}}} = \sqrt{\frac{2R^3}{GM}}$$

This motivates us to define the *dynamical timescale* of the star as

$$\tau_{\mathrm{dyn}} \equiv \sqrt{\frac{R^3}{GM}}. \tag{9}$$

This quantity represents the characteristic timescale over which the star responds to departures from hydrostatic equilibrium. After any perturbation which pushes a star out of hydrostatic equilibrium[4], the star will (if able) come back into hydrostatic equilibrium on a timescale τ_{dyn}. Turning this statement around, when considering physical processes that unfold over timescales *longer* than τ_{dyn}, we can assume that stars remain in almost-perfect hydrostatic equilibrium.

[4] For instance, the gravitational force arising from the nearby passage of another star.

Further Reading

Kippenhahn, Weigert & Weiss, §2.1; *Ostlie & Carroll*, §10.1; *Prialnik*, §2.3.