Spherical Symmetry

Under the assumption of spherical symmetry[1] the physical quantities describing the structure of a star — density, pressure, temperature, etc. — depend only on the radial coordinate r. The center of the star is at $r = 0$, by definition; likewise, remembering that R denotes the stellar radius, the surface of the star is at $r = R$.

[1] For most stars, this is a very good approximation. Exceptions can arise when a star is rotating rapidly, or when it is distorted by the tidal forces of a binary companion; but we'll neglect these complications for now.

Mass Equation

The distribution of matter in a star is described by a pair of connected functions: $\rho(r)$ is the local density at radial coordinate r, while $m(r)$ is the mass of the star contained within the sphere with radius r. This latter quantity is often known as the *interior mass*, to distinguish it from the total stellar mass $M \equiv m(R)$.

The density and interior mass are related by the *mass equation*

$$m(r') = \int_0^{r'} 4\pi r^2 \rho \, \mathrm{d}r, \tag{1}$$

which can also be written in the differential form

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho. \tag{2}$$

Because $\mathrm{d}m/\,\mathrm{d}r$ is always positive, m is a monotonic-increasing function of r. Therefore, there is a one-to-one mapping between interior mass and radial coordinate, and we can use m itself as a coordinate for specifying location within the star. For instance, $m = 0.75\,M$ corresponds to any point on a sphere positioned so that 75% of the star's mass lies inside, and the remaining 25% outside.

Fig. 1 illustrates the density and interior mass functions for a model of the Sun. The matter at the center is around 160 times more dense than water, yet it remains in a gaseous state due to its high temperature. Moving from the center to the surface, the density drops rapidly while the interior mass grows steadily.

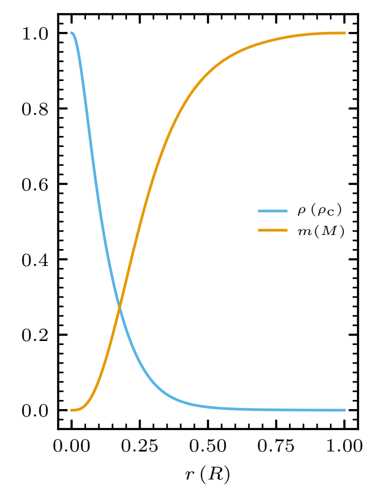

Figure 1: The density ρ (in units of its central value $\rho_c = 160\,\mathrm{g\,cm^{-3}}$) and interior mass m (in units of its surface value $M = M_\odot = 1.989 \times 10^{33}$ g), plotted as a function of radial coordinate r for a *MESA* model of the present-day Sun.

Gravitational Field

Stars are held together by the gravitational field arising from their mass. This field is described by the gravitational potential Φ, which satisfies the *spherical Poisson equation*

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G \rho. \tag{3}$$

Here, G is the universal constant of gravitation[2]. Multiplying both

[2] In cgs units, this has the value $G = 6.674 \times 10^{-8}\,\mathrm{cm\,s^{-2}\,g^{-1}}$.

sides by r^2 and integrating with respect to r, we obtain an expression for the gravitational acceleration (also known just as the *gravity*),

$$g \equiv -\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{Gm}{r^2}. \tag{4}$$

Formally, this can be integrated again to obtain the gravitational potential

$$\Phi(r') = -\int_0^{r'} \frac{Gm}{r^2}\,\mathrm{d}r + C, \tag{5}$$

where the constant C is usually chosen so that $\lim_{r\to\infty}\Phi = 0$. In most cases, however, it is not possible to evaluate this integral analytically.

Fig. 2 illustrates the potential and gravity for the same solar model shown in Fig. 1. The gravity is strongest around $\simeq 0.25\,R$, and drops to zero at the center of the star (where the potential is at its minimum).

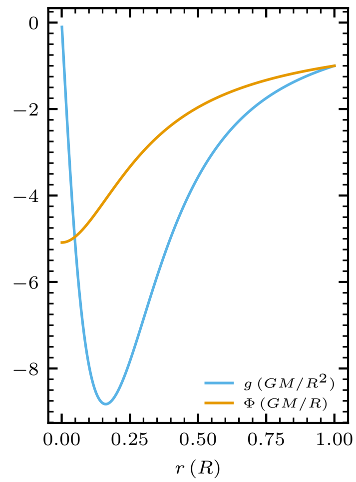

Figure 2: The gravity g (in units of its surface magnitude $GM/R^2 = 2.69\times10^3$ cm s^{-2}) and gravitational potential Φ (in units of its surface magnitude $GM/R = 1.89\times10^{15}$ cm^2 s^{-2}), plotted as a function of radial coordinate r for the *MESA* model of the present-day Sun (cf. Fig. 1).

Gravitational Potential Energy

Imagine building a star layer-by-layer, by bringing each layer down from infinity to its final position. When adding a new layer at coordinates r and m, the gravitational potential is $\Phi = -Gm/r$; therefore, the potential energy of the layer is

$$\mathrm{d}U \equiv \Phi\,\mathrm{d}m = -\frac{Gm}{r}\,\mathrm{d}m, \tag{6}$$

where dm is the layer's mass. Integrating over all layers, we obtain the *gravitational potential energy* of the star

$$U \equiv \int \mathrm{d}U = -\int_0^M \frac{Gm}{r}\,\mathrm{d}m. \tag{7}$$

As with the potential, in most cases it is not possible to evaluate this integral analytically[3]. For simplicity, we will often express the gravitational energy as

$$U = -f_U\frac{GM^2}{R}, \tag{8}$$

where the 'shape factor' f_U is a number of order unity. For the solar model shown in Figs. 1 and 2, $f_U = 1.67$ and the gravitational potential energy is $U = -6.12\times10^{48}$ erg. With a luminosity $L_\odot = 3.83\times10^{33}$ erg s^{-1}, the Sun could survive for 5.07×10^7 yr by converting its gravitational energy into light; this is a long time by human standards, but very short compared to the age of the Sun.

[3] Although the integral looks rather straightforward, don't forget that r in the integrand itself depends on m in some complicated fashion

Further Reading

Kippenhahn, Weigert & Weiss, §§1.1–1.3; *Ostlie & Carroll*, §§2.2,10.1; *Prialnik*, §1.3.