The Two-Body Problem

Consider a pair of particles[1] with masses M_1 and M_2, moving subject to their mutual gravitational attraction. Mathematically, this *two-body problem* involves solving the coupled differential equations that arise from applying Newton's second law and law of gravitation:

[1] In our context, binary stars or a planet orbiting the Sun.

$$M_1 \ddot{\mathbf{r}}_1 = \frac{G M_1 M_2}{r^3}\,\mathbf{r}, \qquad M_2 \ddot{\mathbf{r}}_2 = -\frac{G M_1 M_2}{r^3}\,\mathbf{r}, \tag{1}$$

where $\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ is the displacement from the first particle to the second, $r \equiv |\mathbf{r}|$, and we use the standard notation that a dot indicates differentiation with respect to time.

The One-Body Problem

Combining the two equations above, we can reduce the two-body problem to an equivalent *one-body problem* of a particle moving in a central potential:

$$\mu \ddot{\mathbf{r}} = -\frac{G M_1 M_2}{r^3}\,\mathbf{r}, \tag{2}$$

where

$$\mu \equiv \frac{M_1 M_2}{M_1 + M_2} \tag{3}$$

is the *reduced mass* of the system. It can be shown that solutions $\mathbf{r}(t)$ to the equation of motion (2) lie in a plane. Therefore, we write the equation in terms of polar coordinates[2] (r, θ) in this plane:

[2] Consult any classical mechanics text to understand the conversion of the equation of motion to polar coordinates.

$$\mu \left(\ddot{r} - r\dot{\theta}^2 \right) = -\frac{G M_1 M_2}{r^2} \tag{4}$$

$$\mu \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) = 0 \tag{5}$$

The θ equation is trivially integrated once to obtain

$$\mu r^2 \dot{\theta} = L \tag{6}$$

where L is a constant that we can recognize as the angular momentum of the system. The area swept out by the particle in time interval Δt is $r^2\dot{\theta}\Delta t = (L/m)\Delta t$, and therefore we see that Kepler's second law (*Handout* xxx) arises naturally from conservation of angular momentum.

To solve the r equation, we transform[3] from t to θ as the independent variable, and to $u = 1/r$ as the dependent variable. After some algebra, we obtain

[3] By applying the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta},$$

under the assumption that L is non-zero.

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{G M_1 M_2 \mu}{L^2} \tag{7}$$

The solution to this second-order differential equation is

$$u = \frac{G M_1 M_2 \mu}{L^2}(1 + e\cos\theta) \tag{8}$$

where e is a constant of integration[4]. Hence, the solution to the radial equation is found as

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{9}$$

where

$$a = \frac{L^2}{GM_1M_2\mu}\frac{1}{1-e^2}. \tag{10}$$

For $|e| < 1$ the solution (9) describes an ellipse with semi-major axis a, eccentricity e, and one focus situated at the origin[5].

Suppose the origin of our coordinate system is at the center-of-mass of the system. It follows[6]that the positions of each particle can be calculated from $\mathbf{r}$ as[7]

$$\mathbf{r}_1 = -\frac{M_2}{M_1+M_2}\,\mathbf{r}, \qquad \mathbf{r}_2 = \frac{M_1}{M_1+M_2}\,\mathbf{r}. \tag{11}$$

When we apply these expressions to the Solar System, with the Sun as particle 1 and a planet as particle 2, the fact that $M_1 \gg M_2$ leads to the approximations:

$$\mathbf{r}_1 \approx \mathbf{0}, \qquad \mathbf{r}_2 \approx \mathbf{r}. \tag{12}$$

Hence, we find that the approximate orbit of the planet is an ellipse, with the Sun stationary at one focus; this is Kepler's first law. Of course, a more-precise description is that the orbits of the Sun and a planet are *both* ellipses[8], with the center of mass at one focus.

[4] Because this is the solution to a second-order equation, there should be two constants of integration. However, we've implicitly chosen one to ensure that u is maximal at $\theta = 0$.

[5] For $|e| = 1$, eqn. (9) describes a parabola; and for $|e| > 1$ a hyperbola. However, these are not bound solutions, and so we won't consider them further.

[6] From setting the position coordinate of the center-of-mass, $\mathbf{r}_{\rm COM} \equiv M_1\mathbf{r}_1 + M_2\mathbf{r}_2$, to zero.

[7] The minus sign in the first equation means that the two particles are always on opposite sides of the center-of-mass.

[8] The semi-major axes of these ellipses are not the same:

$$a_1 = \frac{M_2}{M_1+M_2}\,a, \quad a_2 = \frac{M_1}{M_1+M_2}\,a.$$

Note that $a_1 + a_2 = a$.

The Orbital Period

To determine the orbital period P of a two-body system, consider the simple case of a circular [9]binary ($e = 0$). The angular velocity of each particle is uniform, $\dot{\theta} = 2\pi/P$. Combining this with eqns. (6) and (10), we obtain after some algebra the result

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)}a^3. \tag{13}$$

Therefore, the period of the system depends only on combined mass of the particles and the semi-major axis of the system — not on their individual masses. When applied to planets in the Solar System, $M_1 + M_2 \approx M_\odot$, and we find that

$$\left(\frac{P}{1\,\mathrm{yr}}\right)^2 = \left(\frac{a}{1\,\mathrm{au}}\right)^3,$$

which is Kepler's third law.

[9] It can be shown (although with more effort) that the results derived here also apply to eccentric binaries.

Further Reading

Ostlie & Carroll, §2.3; *Prialnik*, §11.1.