Overview

We're now reaching the end of the journey we began in *Handout* IV: we have enough understanding of stellar physics to assemble a closed[1] set of equations that — if we can solve them — will allow us to theoretically predict the structure and evolution of stars. In this handout, we're going to collect these equations in one place (albeit in slightly different forms than when we introduced them), and discuss the boundary conditions that accompany them.

First, let us define the variables that we'll be using. The independent variables are the interior mass[2] m and the time t. The dependent variables are the radial coordinate r, interior luminosity ℓ, density ρ and temperature T, plus a set of mass fractions $\{\mathcal{X}_k\}$ for the elements.

[1] In the sense that there are as many equations as unknowns, so we have hope for finding a well-determined solution.

[2] We use m rather than r because the stellar mass M generally remains unchanged throughout a star's evolution, but the stellar radius R does not.

Structure Equations

The equations of stellar structure are the first-order *partial*[3] differential equations describing conservation of mass, momentum and energy, plus an accompanying differential equation governing energy transport; in order,

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}, \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}, \tag{2}$$

$$\frac{\partial \ell}{\partial m} = \epsilon_{\rm nuc} - \epsilon_{\nu} - \frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}, \tag{3}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla_T. \tag{4}$$

The last equation[4] must be augmented with a formula for the dimensionless temperature gradient ∇_T. Under the simplifying assumption that convection is everywhere efficient, we can write

$$\nabla_T = \begin{cases} \nabla_{\rm ad} & \text{if } \nabla_{\rm rad} > \nabla_{\rm ad} \quad \text{(convective)} \\ \nabla_{\rm rad} & \text{if } \nabla_{\rm rad} < \nabla_{\rm ad} \quad \text{(radiative)} \end{cases}, \tag{5}$$

where $\nabla_{\rm rad}$ is defined in eqn. (8) of *Handout* XII, and $\nabla_{\rm ad}$ in eqn. (6) of *Handout* XIII.

[3] Partial, because we're explicitly accounting for the fact that the stellar structure depends on time as well as space.

[4] This equation may look unfamiliar, but it is simply a different way of expressing the definition of ∇_T (see eqn. 7 of *Handout* XII).

Evolution Equations

To establish how a star changes with time, we augment the four structure equations (1–4) with a set of evolution equations that describe how the mass fractions $\{\mathcal{X}_k\}$ change with time. For element j, the evolution equation[5] is

[5] In convection zones, we must modify this equation with additional terms accounting for the rapid mixing that occurs; often, however, it is simpler to assume that convection zones remain fully mixed, so that $\partial \mathcal{X}_j / \partial m = 0$ over the extent of the zone for all j.

$$\frac{\partial \mathcal{X}_j}{\partial t} = \frac{\mathcal{A}_j \mathrm{m_H}}{\rho} \sum_{k \neq j} \left(r_{kj} - r_{jk} \right), \qquad (6)$$

where r_{kj} represents the rate (in particles per unit volume) at which the element is created from another element k via nuclear reactions.

Constitutive Relations

The structure and evolution equations are accompanied by a set of *constitutive relations* that specify how the properties of stellar material depend on temperature, density and composition. These are the equation of state giving P; the internal energy equation giving u; the opacity equation giving κ; the nuclear reaction equations giving r_{kj} and ϵ_{nuc}; and the neutrino loss equation giving ϵ_ν.

Boundary Conditions

To complete our specification of the overall problem, we augment the four structure equations (1–4) with four boundary conditions. The boundary conditions at the center are

$$\left.\begin{array}{l} \ell \to 0 \\ r \to 0 \end{array}\right\} \quad \text{as } m \to 0, \qquad (7)$$

For the surface boundary conditions, we use a formula[6] giving the temperature in the photosphere as a function of optical depth as

$$T^4(\tau) = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3} \right). \qquad (8)$$

[6] This formula comes from the theory of stellar atmospheres, adopting the *Eddington-gray* approximation. T_{eff} is the usual effective temperature defined by the luminosity L and radius R at the outer boundary (see eqn. 8 of *Handout* II).

To find the pressure, we integrate the equation of hydrostatic equilibrium down from the top of the photosphere. If we assume that the opacity and gravity remain constant throughout the photosphere, then

$$P(\tau) = \frac{GM}{R^2 \kappa} \tau. \qquad (9)$$

Usually, we define the nominal outer boundary to be where $T = T_{\mathrm{eff}}$; from the temperature equation, we see this corresponds to $\tau = 2/3$. Therefore, evaluating the pressure at this optical depth, we find the outer boundary conditions as

$$\left.\begin{array}{l} P \to \dfrac{GM}{R^2 \kappa} \dfrac{2}{3} \\ T \to T_{\mathrm{eff}} \end{array}\right\} \quad \text{as } m \to M. \qquad (10)$$

Further Reading

Kippenhahn, Weigert & Weiss, §10.1; *Ostlie & Carroll*, §10.5.