Free Electrons from Ionization

In previous handouts we've discussed the important role that free electrons play in determining the properties of stellar material; and we've derived expressions[1] for how their pressure $P_{\rm el}$ depends on the temperature T and their number density $n_{\rm el}$. However, we've yet to delve into the physical process responsible for determining $n_{\rm el}$: *ionization*.

[1] Refer to *Handout* XVI, specifically eqn. (7) in the classical limit, and eqn. (11) in the completely degenerate limit.

As stellar material becomes progressively hotter, some of the thermal energy can be used to unbind electrons from atoms, adding them to the pool of free electrons. In this handout we'll explorer this process for the simplest case of hydrogen.

Ionization of Hydrogen

The ionization of hydrogen can be written as a chemical process involving three particles:

$$\mathrm{H} \leftrightarrow \mathrm{H}^{+} + e^{-} \tag{1}$$

On the left-hand side, we have a hydrogen atom; and on the right-hand side we have a hydrogen ion[2] and a free electron. Like all chemical processes, this process is bidirectional; in a sample of stellar material, there are left-to-right reactions (ionizations) taking place at the same time as right-to-left reactions (recombination). Any difference between the rates of these two reactions will cause the relative numbers of atoms, ions and free electrons to change over time; but eventually and equilibrium will be reached where the two reaction rates are balanced.

[2] Which is, of course, just a bare proton

The Saha Equation for Hydrogen

To determine the relative numbers of hydrogen atoms, ions and free electrons in this equilibrium, we start by writing down the momentum distribution functions[3] for the three particle types[4] :

[3] See *Handout* XVI for a reminder of what a momentum distribution function is, and how we can use it.

[4] Here, the subscripts '0' and '1' refer to the hydrogen atoms and ions, respectively; while 'el' refers to free electrons, as usual.

$$\begin{aligned}
f_0(p) &= \frac{8\pi p^2}{h^3} \frac{1}{\exp[p^2/(2\mathrm{m_H} k_\mathrm{B} T) - \chi/(k_\mathrm{B}T) - \psi_0] + 1}, \\
f_1(p) &= \frac{1}{2}\frac{8\pi p^2}{h^3} \frac{1}{\exp[p^2/(2\mathrm{m_H} k_\mathrm{B} T) - \psi_1] + 1} \\
f_\mathrm{el}(p) &= \frac{8\pi p^2}{h^3} \frac{1}{\exp[p^2/(2\mathrm{m_{el}} k_\mathrm{B} T) - \psi_\mathrm{el}] + 1}
\end{aligned} \tag{2}$$

In the distribution function for the atoms, an extra term $-\chi/k_\mathrm{B}T$ appears in the exponent; χ is the ionization energy[5], and this term accounts for the (negative) potential energy of the bound electron.

[5] For hydrogen, $\chi = 13.6\,\mathrm{eV} = 2.18 \times 10^{-11}\,\mathrm{erg}$.

In the distribution function for the ions, the initial factor of $1/2$ is to avoid double counting the spins of the ions and free electrons[6]. As a final remark, we've assumed (as a simplifying approximation) that both atoms and ions have a mass $\mathrm{m_H}$.

[6] For each ion and free electron *pair*, there are two spin states: aligned and anti-aligned. The $8\pi p^2/h^3$ factor in the distribution function already includes two spin states, and so either the ion or free electron distribution function must include a factor of $1/2$ to avoid double counting.

Let's assume that all particles are in the classical limit, so that the degeneracy parameters are all large and negative. Then, we can neglect the $+1$ term in the denominator of the distribution functions. Integrating over momentum, we arrive at expressions for the number density of each type of particle:

$$\begin{aligned} n_{\mathrm{H},0} &= \frac{2(2\pi \mathrm{m_H} k_{\mathrm{B}} T)^{3/2}}{h^3} \exp[\psi_0 + \chi/(k_{\mathrm{B}} T)], \\ n_{\mathrm{H}}1 &= \frac{(2\pi \mathrm{m_H} k_{\mathrm{B}} T)^{3/2}}{h^3} \exp[\psi_1], \\ n_{\mathrm{el}} &= \frac{2(2\pi \mathrm{m_{el}} k_{\mathrm{B}} T)^{3/2}}{h^3} \exp[\psi_{\mathrm{el}}]. \end{aligned} \tag{3}$$

As a next step, we take advantage of the fact that the degeneracy parameters are not completely independent; the reaction given in eqn. (1) means that[7]

$$\psi_0 = \psi_1 + \psi_{\mathrm{el}}. \tag{4}$$

[7] This is a result from statistical mechanics, that arises due to chemical equilibrium: for every hydrogen atom destroyed, there must be a hydrogen ion and free electron created.

This allows us to combine the three equations for the number density into a single equation, known as the *Saha equation*[8] for hydrogen:

$$\frac{n_{\mathrm{H},1}\, n_{\mathrm{el}}}{n_{\mathrm{H},0}} = \frac{(2\pi \mathrm{m_{el}} k_{\mathrm{B}} T)^{3/2}}{h^3} \exp[-\chi/(k_{\mathrm{B}} T)]. \tag{5}$$

[8] After the Indian physicist Meghnad Saha, who first derived it.

To solve this equation, we must augment it with a couple of other relations. Conservation of baryon number and charge require that,

$$n_{\mathrm{H},0} + n_{\mathrm{H},1} = n_{\mathrm{H}}, \qquad n_{\mathrm{H},1} = n_{\mathrm{el}}, \tag{6}$$

respectively, where n_{H} is the total number density of hydrogen (which we assume is a known quantity).

The Ionization Fraction

Combining the Saha equation (5) with the conservation relations following it, we can calculate the ionization fraction $x \equiv n_1/n_{\mathrm{H}}$ of the hydrogen — given T and n_{H} — by solving the quadratic equation

$$\frac{x^2}{1-x} = \frac{(2\pi \mathrm{m_{el}} k_{\mathrm{B}} T)^{3/2}}{n_{\mathrm{H}} h^3} \exp[-\chi/(k_{\mathrm{B}} T)]. \tag{7}$$

Fig. 1 demonstrates the application of this equation.

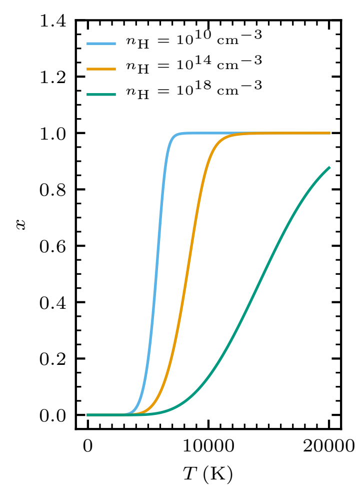

Figure 1: The hydrogen ionization fraction x, plotted as a function of temperature T for three choices of the hydrogen number density n_{H}. For the lower densities, note how almost complete ionization is achieved, even though $\chi/k_{\mathrm{B}}T \ll 1$; this is a consequence of the $(2\pi \mathrm{m_{el}} k_{\mathrm{B}} T)^{3/2}/n_{\mathrm{H}}$ term in eqn. (7).

Further Reading

Kippenhahn, Weigert & Weiss, §14; *Ostlie & Carroll*, §8.1; *Prialnik*, §3.6.