The Distribution Function

So far we've been describing the properties of stellar material through an equation of state (see, e.g., *Handout* VI and *Handout* XV). However, a more-fundamental representation is in terms of the *momentum distribution function*[1] $f(p)$. Given this function, we can evaluate the number density n and pressure P via the integrals

$$n = \int_0^\infty f(p)\, \mathrm{d}p \tag{1}$$

and

$$P = \frac{1}{3} \int_0^\infty f(p)\, p\, v(p)\, \mathrm{d}p, \tag{2}$$

where $v(p)$ is the velocity of a particle with momentum p.

[1] The product $f(p)\,\mathrm{d}p$ gives the number of particles per unit volume with momenta between p and $p + \mathrm{d}p$.

The Free Electron Distribution Function

Because free electrons are *fermions*[2], they follow the Fermi-Dirac distribution function

$$f(p) = f_{\mathrm{el}}(p) \equiv \frac{8\pi p^2}{h^3} \frac{1}{\exp[p^2/(2\mathrm{m_{el}} k_\mathrm{B} T) - \psi] + 1}, \tag{3}$$

where h is Planck's constant, $\mathrm{m_{el}}$ the electron mass, and ψ the *degeneracy parameter*, which quantifies the extent to which the Pauli exclusion principle affects the electrons' behavior. Typically, we use eqn. (1) to determine ψ from n_{el} and T; and then apply eqn. (2) to evaluate the electron pressure.

[2] Particles with half-integer spins, which are subject to the Pauli exclusion principle

Free Electrons in the Classical Limit

In the *classical limit*, where ψ is large and negative, the free electron distribution function is well approximated by

$$f_{\mathrm{el}}(p) \approx \frac{8\pi p^2}{h^3} \exp[-p^2/(2\mathrm{m_{el}} k_\mathrm{B} T)] \exp[\psi] \tag{4}$$

We can solve[3] for $\exp[\psi]$ in terms of n_{el} and T, to obtain

$$\exp[\psi] = \frac{h^3 n_{\mathrm{el}}}{2(2\pi m k_\mathrm{B} T)^{3/2}}. \tag{5}$$

Combining this with the preceding equation, we find

$$f_{\mathrm{el}}(p) \approx \frac{4\pi p^2 n_{\mathrm{el}}}{(2\pi \mathrm{m_{el}} k_\mathrm{B} T)^{3/2}} \exp[-p^2/(2\mathrm{m_{el}} k_\mathrm{B} T)]. \tag{6}$$

This can be recognized as a form of the well-known *Maxwell-Boltzmann* distribution function for ideal gasses (see Fig. 1). Substituting it into

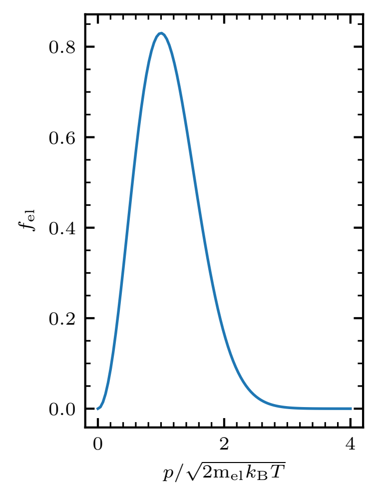

Figure 1: The electron momentum distribution function f_{el}, plotted as a function of scaled momentum $p/\sqrt{2\mathrm{m_{el}} k_\mathrm{B} T}$ in the classical (Maxwell-Boltzmann) limit. The vertical scale has been chosen so that the area under the curve is unity.

[3] To do this for yourself, first substitute the expression for $f_{\mathrm{el}}(p)$ into the integral (1) for n_{el}.

eqn. (2), and assuming the electrons are non-relativistic[4], the electron pressure follows as

$$P_{\rm el} = n_{\rm el} k T. \tag{7}$$

This is, of course, the ideal-gas EOS for free electrons!

[4] So that $v(p) = p/\mathrm{m}_{\rm el}$.

Free Electrons in the Completely Degenerate Limit

In the *completely degenerate* limit, where ψ is large and positive, the free electron distribution function is well approximated by

$$f_{\rm el}(p) \approx \begin{cases} \frac{8\pi p^2}{h^3} & p < p_{\rm F}, \\ 0 & p > p_{\rm F} \end{cases} \tag{8}$$

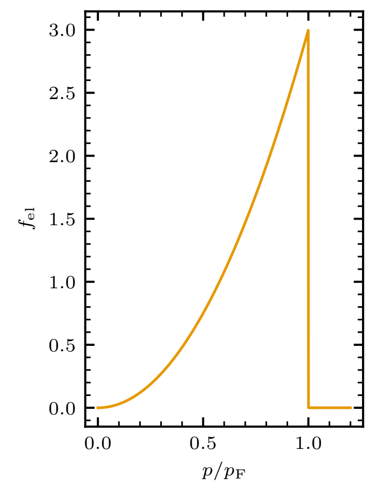

Figure 2: The electron momentum distribution function $f_{\rm el}$, plotted as a function of scaled momentum $p/p_{\rm F}$ in the completely degenerate limit. As with Fig. 1, the vertical scale has been chosen so that the area under the curve is unity. Note the abrupt cutoff in $f_{\rm el}$ for momenta $p > p_{\rm F}$.

(see Fig. 2). Here, $p_{\rm F}$ is the Fermi momentum, defined when $\psi \gg 1$ by

$$\frac{p_{\rm F}^2}{2\mathrm{m}_{\rm el} k_{\rm B} T} = \psi. \tag{9}$$

We can solve for $p_{\rm F}$ in terms of $n_{\rm el}$ to obtain

$$p_{\rm F} = \left(\frac{3h^3 n_{\rm el}}{8\pi} \right)^{1/3}, \tag{10}$$

which defines the Fermi momentum at any density. Again assuming that the electrons are non-relativistic, the electron pressure follows as

$$P_{\rm el} = \left(\frac{1}{3} \right)^{2/3} \frac{h^2}{20\mathrm{m}_{\rm el}} n_{\rm el}^{5/3} \tag{11}$$

This is a significant result: it tells us that the pressure of completely degenerate electrons is independent of temperature, and scales as the 5/3 power of the density.

Partial Degeneracy

For the intermediate case of *partial degeneracy*[5], we have to evaluate $P_{\rm el}$ numerically. However, it's useful to get a sense of what parts of parameter space correspond to the classical and degenerate limits. This can be done by comparing the Fermi energy $\mathcal{E}_{\rm F} \equiv p_{\rm F}^2/2\mathrm{m}_{\rm el}$ against the Boltzmann energy $k_{\rm B} T$:

$$\mathcal{E}_{\rm F} \ll k_{\rm B} T \rightarrow \text{classical}, \qquad \mathcal{E}_{\rm F} \gg k_{\rm B} T \rightarrow \text{degenerate}. \tag{12}$$

[5] I.e., when $|\psi|$ is not large.

Thus, degeneracy is important either when the temperature is low, or the Fermi energy (and therefore the density) is high.

Further Reading

Kippenhahn, Weigert & Weiss, §15; *Ostlie & Carroll*, §16.3; *Prialnik*, §3.3.