General Equation of State

In *Handout* VI, we wrote down the general equation of state (EOS) for stellar material as

$$P = P_{\rm ion} + P_{\rm el} + P_{\rm rad} \tag{1}$$

where the terms on the right-hand side represent the partial pressures from ions, free electrons and radiation, respectively. In this handout we revisit this expression to flesh out these terms in greater detail.

Radiation Pressure

Radiation pressure arises because photons have momentum and therefore can exert a force. Its form is very simple:

$$P_{\rm rad} = \frac{aT^4}{3}, \tag{2}$$

where T is the temperature and a is the usual radiation constant.

Generally, the radiation pressure becomes important in stars with masses $M \gtrsim 10\,{\rm M}_\odot$; hence, we have been well justified in neglecting it when dealing with the evolution, e.g., of the Sun.

Ion Pressure

The ions[1] in stellar material can (almost) always be treated like an ideal gas. The partial pressure associated with ions of isotope j is then given by

$$P^j_{\rm ion} = n^j_{\rm ion} k_{\rm B} T = \frac{\rho \mathcal{X}_j}{\mathcal{A}_j {\rm m_H}} k_{\rm B} T \tag{3}$$

[1] Remember from *Handout* VI that we are using 'ions' generically to denote atoms in any ionization state.

Here, $n^j_{\rm ion}$ in the first equality is the number density of the isotope. In the second equality, we re-write this number density in terms of the overall mass density ρ of the stellar material, together with the mass fraction $\mathcal{X}_j$ and the mass number[2] $\mathcal{A}_j$ of the isotope. Summing over all isotopes, we obtain the total ion pressure as

$$P_{\rm ion} = \frac{\rho}{\rm m_H} \left[\sum_j \frac{\mathcal{X}_j}{\mathcal{A}_j} \right] k_{\rm B} T. \tag{4}$$

[2] The number of protons plus neutrons in the nucleus. Note that eqn. (3) treats the protons and neutrons as if they have the same mass $\rm m_H$ as the hydrogen atom — not exactly true, but also not too bad of an approximation.

Electron Pressure

Unlike the ions, the fee electrons in stellar material don't always behave like an ideal gas; at (relatively) low temperatures and high densities, quantum mechanical effects cause their EOS to behave in a very non-ideal manner. We'll look into these effects shortly; but for now let's assume ideal behavior.

Suppose that each ion of isotope j is associated with an average of $\mathcal{N}_j$ free electrons[3]. The pressure from these free electrons is

[3] Another way of saying this is that the average charge per ion is $\mathcal{N}_j\, e$, where e is the fundamental charge unit.

$$P_{\rm el}^{j} = \mathcal{N}_j P_{\rm ion}^{j} = \mathcal{N}_j \frac{\rho \mathcal{X}_j}{\mathcal{A}_j \mathrm{m_H}} k_{\rm B} T. \tag{5}$$

Summing over all isotopes, we obtain the total electron pressure as

$$P_{\rm el} = \frac{\rho}{\mathrm{m_H}} \left[\sum_j \frac{\mathcal{N}_j \mathcal{X}_j}{\mathcal{A}_j} \right] k_{\rm B} T. \tag{6}$$

Ions & Electrons Together

When we can neglect radiation pressure, and treat the free electrons as an ideal gas, we can combine equations (4) and (6) to obtain

$$P = \frac{\rho}{\mathrm{m_H}} \left[\sum_j \frac{(1+\mathcal{N}_j)\mathcal{X}j}{\mathcal{A}_j} \right] k_{\rm B} T. \tag{7}$$

Comparing this against the ideal-gas EOS given in eqn. (4) of *Handout* VI, we can see that the quantity in square brackets must be equivalent to the reciprocal of the mean molecular weight μ:

$$\mu \equiv \left[\sum_j \frac{(1+\mathcal{N}_j)\mathcal{X}j}{\mathcal{A}_j} \right]^{-1}. \tag{8}$$

At first glance, this seems to be quite different than the expressions for μ given in *Handout* VI. However, consider the neutral limit where $\mathcal{N}_j \to 0$ for all j; then, the mean molecular weight can be written as

$$\mu = \left[\sum_j \frac{\mathcal{X}j}{\mathcal{A}_j} \right]^{-1} \approx \left[X + \frac{Y}{4} + \frac{Z}{12} \right]^{-1}, \tag{9}$$

where the second, approximate equality follows from assuming[4] $\mathcal{A}_j \approx 12$ for all the metals. This result is the same as eqn. (6) of *Handout* VI.

In the opposite, fully ionized limit, we have $\mathcal{N}_j \to \mathcal{Z}_j$, where $\mathcal{Z}_j$ is the atomic number[5] of isotope j. Then, the mean molecular weight can be written as

$$\mu = \left[\sum_j \frac{(1+\mathcal{Z}_j)\mathcal{X}j}{\mathcal{A}_j} \right]^{-1} \approx \left[2X + \frac{3Y}{4} + \frac{Z}{2} \right]^{-1}, \tag{10}$$

where the second, approximate equality follows from assuming[6] $\mathcal{Z}_j + 1 \approx \mathcal{A}_j/2$ for all the metals. This result is the same as eqn. (7) of *Handout* VI.

[4] Based on the recognition that ^{12}C is usually the most abundant of the light metals.

[5] The number of protons in the nucleus.

[6] Based on the fact that the number of neutrons, for metals, is about the same as the number of protons.

Further Reading

Kippenhahn, Weigert & Weiss, §4.2; *Ostlie & Carroll*, §10.2; *Prialnik*, §§3.2-3.4.