An Algorithm for Convection

At any point within a star, suppose we know the opacity κ, the interior luminosity ℓ and the thermodynamic state (T, P and ρ) of the material. How then do we calculate the temperature gradient ∇_T? Putting together our knowledge from previous handouts, we can devise a suitable algorithm. First, evaluate $\nabla_{\rm rad}$ from κ, ℓ and the state (see eqn. 8 of *Handout* XII), and $\nabla_{\rm ad}$ from the state (see eqn. 6 of *Handout* XIII). Then, compare these two quantities[1] and set up ∇_T as follows:

$$\nabla_T = \begin{cases} \nabla_{\rm rad} & \text{if } \nabla_{\rm rad} < \nabla_{\rm ad}, \\ \varphi_{\rm conv}\nabla_{\rm ad} + (1-\varphi_{\rm conv})\nabla_{\rm rad} & \text{if } \nabla_{\rm rad} > \nabla_{\rm ad}. \end{cases} \qquad (1)$$

[1] Note that we're comparing $\nabla_{\rm rad}$ against $\nabla_{\rm ad}$; contrast with the Schwarzschild criterion (eqn. 8 of *Handout* XIII), which relates ∇_T and $\nabla_{\rm ad}$.

Here, $\varphi_{\rm conv}$ as a *convective efficiency parameter*, which we'll discuss below.

The upper choice in eqn. (1) satisfies the Schwarzschild criterion $\nabla_T < \nabla_{\rm ad}$, indicating that no convection occurs. Then, radiation transports all of the interior luminosity: $\ell_{\rm rad} = \ell$ and $\ell_{\rm conv} = 0$. By contrast, the lower choice violates the Schwarzschild criterion, indicating that convection must be present. Then, the interior luminosities transported by radiation and convection are

$$\frac{\ell_{\rm rad}}{\ell} = \frac{\varphi_{\rm conv}\nabla_{\rm rad} + (1-\varphi_{\rm conv})\nabla_{\rm ad}}{\nabla_{\rm rad}}, \qquad \frac{\ell_{\rm conv}}{\ell} = \varphi_{\rm conv}\frac{\nabla_{\rm rad}-\nabla_{\rm ad}}{\nabla_{\rm rad}}. \qquad (2)$$

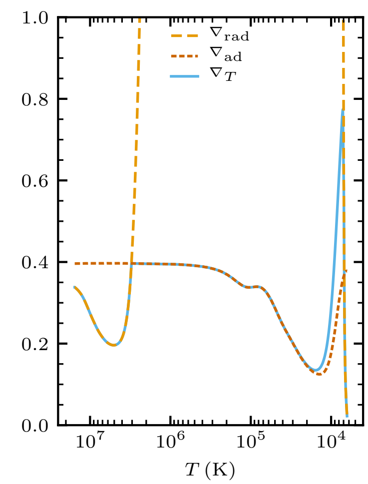

Figure 1: The dimensionless (∇_T), radiative ($\nabla_{\rm rad}$) and adiabatic ($\nabla_{\rm ad}$) temperature gradients, plotted as a function of temperature T for a *MESA* model of the present-day Sun. The solar convection zone corresponds to the region where $\nabla_{\rm rad} > \nabla_{\rm ad}$.

Convective Efficiency

The efficiency parameter $\varphi_{\rm conv}$ is bounded[2] to lie in the interval $[0,1]$. It depends in a complicated fashion on κ, ℓ and other local quantities; to evaluate it, we need to apply so-called *mixing-length theory* (MLT). We won't go into MLT in detail here, but let's consider a couple of limiting cases. In the 'efficient' limit $\varphi_{\rm conv} \rightarrow 1$, convection transports as much energy as it is able, and the radiative and convective luminosities are

$$\frac{\ell_{\rm rad}}{\ell} \rightarrow \frac{\nabla_{\rm ad}}{\nabla_{\rm rad}}, \qquad \frac{\ell_{\rm conv}}{\ell} \rightarrow \frac{\nabla_{\rm rad}-\nabla_{\rm ad}}{\nabla_{\rm rad}}. \qquad (3)$$

[2] This is because $\nabla_{\rm ad} < \nabla_T < \nabla_{\rm rad}$ in convection zones; ∇_T must exceed $\nabla_{\rm ad}$ in order for the convection to occur; but cannot exceed $\nabla_{\rm rad}$ because, otherwise, $\ell_{\rm rad}$ would be larger than ℓ.

In the opposite 'inefficient' limit $\varphi_{\rm conv} \rightarrow 0$, convection transports essentially no energy, and the luminosities are

$$\ell_{\rm rad} \rightarrow \ell, \qquad \ell_{\rm conv} \rightarrow 0. \qquad (4)$$

In stars, it's typically the case that convection is only inefficient ($\varphi_{\rm conv} \rightarrow 0$) when it occurs very near the stellar surface; otherwise, it's efficient. Fig. 1 demonstrates this by plotting the three gradients (∇_T, $\nabla_{\rm rad}$ and $\nabla_{\rm ad}$) for a model of the Sun. Throughout most of the zone where $\nabla_{\rm rad} > \nabla_{\rm ad}$, the high efficiency of convection ($\varphi_{\rm conv} \approx 1$) means that $\nabla_T \approx \nabla_{\rm ad}$.

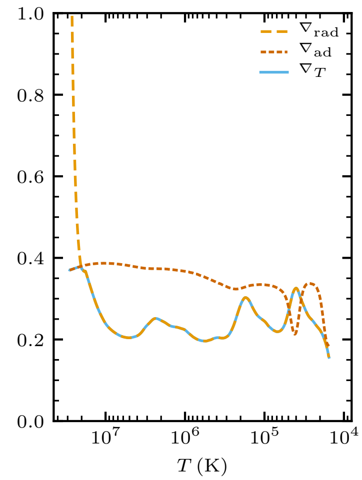

Figure 2: As with Fig. 1, except that a $5\,M_\odot$ star with a core hydrogen abundance $X_{\rm c} \approx 0.38$ is shown (this abundance is chosen to match that of the solar model shown in Fig. 1).

Convection on the Main Sequence

Intermediate-mass ($0.4\,M_\odot \lesssim M \lesssim 1.4\,M_\odot$) stars on the main sequence share the same general structural layout as the Sun (see Fig. 1), comprising a radiative core and a convective envelope. For higher-mass stars ($M \gtrsim 1.4\,M_\odot$) on the main sequence the configuration is opposite, comprising a convective core and a radiative envelope (cf. Fig. 2). Whereas, low-mass stars ($M \lesssim 0.4\,M_\odot$) are convective throughout their entire interior.

To understand these different outcomes, let's examine what can cause $\nabla_{\rm rad}$ to rise above the typical threshold $\nabla_{\rm ad} \approx 2/5$ for the onset of convection. From *Handout* xii, the radiative temperature gradient is

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa \ell P}{mT^4}.$$

The two important terms on the right-hand side are the opacity κ and the interior luminosity-to-mass ratio ℓ/m; if either of these are large, then so too is $\nabla_{\rm rad}$ and convection must occur.

Fig. 3 plots these terms for the solar model, the $5\,M_\odot$ model, and for a $0.2\,M_\odot$ main-sequence model. Breaking down the data for each mass,

- the solar model shows a large opacity in the envelope (due mainly to bound-free and free-free absorption), explaining why convection occurs there;
- the low-mass model shows a large opacity in the envelope *and* the core (again due to bound-free and free-free absorption), explaining why the entire star is convective;
- the high-mass model shows a relatively small opacity everywhere, compared to the other models, due to its lower density. However, the model has a large luminosity-to-mass ratio in the core, explaining why convection occurs there.

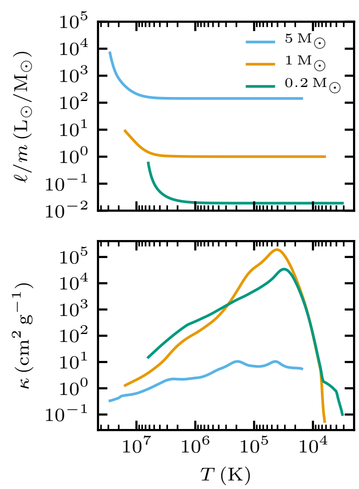

Figure 3: The interior mass-to-luminosity ℓ/m (in units of $L_\odot/M_\odot$) and the opacity, plotted as a function of temperature for *MESA* models with masses $5, M_\odot$, $1\,M_\odot$ and $0.2\,M_\odot$ (the first two are the same models shown in Figs. 1 and 2; the third has the same core hydrogen abundance $X_{\rm c} \approx 0.38$ as the first two).

Further Reading

Kippenhahn, Weigert & Weiss, §§7,22.3; *Ostlie & Carroll*, §10.4; *Prialnik*, §9.2.