Stability of Hydrostatic Equilibrium

Consider a small blob of material with volume ΔV, at rest within a star in hydrostatic equilibrium. Suppose we displace this blob in the radial direction. The blob must expand or contract to remain in pressure equilibrium with its surroundings, and as a result its density will change. Let's denote this change as $\delta\rho_{\rm b}$, and the corresponding change in the density of the blob's surroundings (caused by the blob being at a different position in the star) as $\delta\rho_{\rm s}$[1]. Generally, $\rho_{\rm b} \neq \rho_{\rm s}$, and so there will be a net buoyant force on the blob given[2]by

$$f_{\rm b} = (\delta\rho_{\rm s} - \delta\rho_{\rm b}) g \, \Delta V. \tag{1}$$

Depending on the sign of $f_{\rm b}$, this force will either pull the blob back to its original position, or push it further away. These two outcomes correspond, respectively, to the hydrostatic equilibrium being *stable* or *unstable*.

In the unstable case, any tiny departures from perfect hydrostatic balance in the star will be amplified over time. However, the final outcome of this instability isn't complete disruption of the star, but a steady-state system of circulatory currents, with low-density upwellings balanced by high-density downwellings; these currents are what we have been calling *convection*. When averaged over many circulation timescales[3], hydrostatic balance (eqn. 6 of *Handout* v) still applies.

[1] Here, we use the subscript 'b' to refer to the blob, and the subscript 's' to refer to the blob's surroundings.

[2] This equation comes from applying Archimedes principle:*Any object, totally or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.*

[3] These timescales can vary from minutes to centuries, but are typically much shorter than stellar evolutionary timescales.

The Schwarzschild Criterion

To establish the circumstances leading to convection, let's figure out how we can evaluate the $\delta\rho$ terms appearing in eqn. (1). Regarding the density as a function of pressure and temperature, we can write

$$\delta\rho = \left(\frac{\partial\rho}{\partial T}\right)_P \delta T + \left(\frac{\partial\rho}{\partial P}\right)_T \delta P. \tag{2}$$

(we've dropped subscripts for the moment, because this expression applies separately to both blob and surroundings). Using the ideal-gas equation of state (eqn. 4 of *Handout* vi) to evaluate the partial derivatives[4], this becomes

$$\delta\rho = \rho\left(-\frac{\delta T}{T} + \frac{\delta P}{P}\right). \tag{3}$$

For the surroundings, the temperature change $\delta T_{\rm s}$ can be expressed in terms of the pressure change $\Delta P_{\rm s}$ via

$$\frac{\delta T_{\rm s}}{T} = \left(\frac{{\rm d}\ln T}{{\rm d}\ln P}\right)\frac{\delta P_{\rm s}}{P} = \nabla_T \frac{\delta P_{\rm s}}{P}, \tag{4}$$

[4] Although our derivation here assumes an idea gas EOS, the resulting stability criterion (eqn. 8) is in fact generally applicable to any EOS.

where the second equality invokes the dimensionless temperature gradient defined in eqn. (7) of *Handout* XII.

For the blob, we have to consider how its state adjusts as it expands or contracts. Because the flow of energy is such a slow process in the stellar interior, it's reasonable to assume that this adjustment is adiabatic. Therefore, we write

$$\frac{\delta T_{\rm b}}{T} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad} \frac{\delta P_{\rm b}}{P} = \nabla_{\rm ad} \frac{\delta P_{\rm b}}{P}. \tag{5}$$

The second equality introduces the *adiabatic temperature gradient*

$$\nabla_{\rm ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P}\right)_{\rm ad}, \tag{6}$$

where the subscript 'ad' indicates an adiabatic change. An important distinction between ∇_T and $\nabla_{\rm ad}$ is that the former depends on the temperature stratification $T(P)$ within the star; whereas the latter depends only on the equation of state[5]. This distinction is manifest in the different types of derivatives appearing in these gradients' definitions: ordinary for ∇_T, partial for $\nabla_{\rm ad}$.

Let's now put everything together. Using eqns. (3–5) to evaluate the density changes in the blob and surroundings, the buoyant force becomes

$$f_{\rm b} = (\nabla_{\rm ad} - \nabla_T)\,\frac{\delta P_{\rm b}}{P}\,\rho\, g\, \Delta V, \tag{7}$$

where we've used the fact that $\delta P_{\rm b} = \delta P_{\rm s}$ because the blob remains in pressure equilibrium with its surroundings. For stability, the term in parentheses must be positive[6], leading to the criterion

$$\nabla_T < \nabla_{\rm ad}. \tag{8}$$

This is known as the *Schwarzschild* [7]*criterion* for convective stability. At any point in this star, if this inequality holds then the material is stable against small displacements and there will be no convection. Conversely, if this inequality is violated then the material is unstable and convection will spontaneously commence.

Fig. 1 plots ∇_T and $\nabla_{\rm ad}$ for a model of the Sun. In the inner part of the star ($r \lesssim 0.7\,R$) the Schwarzschild criterion (8) is satisfied and there is no convection. Conversely, in the outer part ($0.7\,R \lesssim r < R$) the criterion is violated and so convection occurs. Note that $\nabla_{\rm ad} \approx 2/5$ throughout most of the star, confirming that the material is well approximated as an ideal gas with $\gamma = 5/3$.

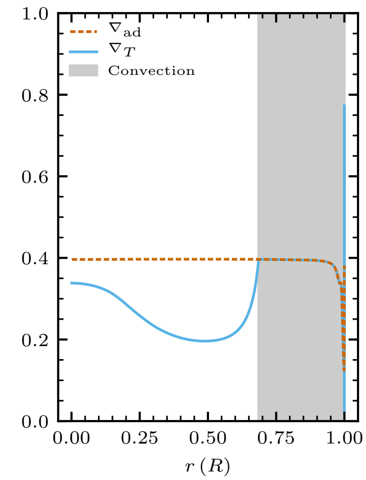

Figure 1: The dimensionless temperature gradient ∇_T and adiabatic temperature gradient $\nabla_{\rm rad}$, plotted as a function of radial coordinate r for a *MESA* model of the present-day Sun. The shading shows the convective region where the Schwarzschild stability criterion $\nabla_T < \nabla_{\rm ad}$ is violated.

[5] For the specific case of an ideal gas EOS, $\nabla_{\rm ad} = (\gamma - 1)/\gamma$, where γ is the usual ratio of specific heats.

[6] To see this, consider an upward displacement of the blob. Because pressure decreases outward in a star in hydrostatic equilibrium, $\Delta P_{\rm s} < 0$. In order for the force to pull the blob back down, $f_{\rm b} < 0$; therefore, the term in parentheses must be positive.

[7] After Martin Schwarzschild, one of the pioneers of modern stellar astrophysics. Martin's *father*, Karl, gave his name to the the radius of the event horizon of black holes.

Further Reading

Kippenhahn, Weigert & Weiss, §6.1; *Ostlie & Carroll*, §10.4; *Prialnik*, §6.5.