Homework Assignment 13 — Solutions

- Q11.1 To confirm that the changes in the effective temperature of the Sun are consistent with its variations in radius and luminoisity, as graphed in Fig. 11.1 of *Ostlie & Carroll*, we can check that the equation
$$L = 4\pi R^2 \sigma T_{\rm eff}^4$$
is satisfied. Let's do this at the start of the graph, the end, and in the middle (at the present age):

Age (10^9 yr)	$L(L_\odot)$	$R(R_\odot)$	$T_{\rm eff}$(K)	$4\pi R^2 \sigma T_{\rm eff}^4 (L_\odot)$
0.00	0.68	0.87	5620	0.68
4.55	1.00	1.00	5777	1.00
8.00	1.36	1.16	5781	1.35

The close agreement between the second and fifth columns confirms that the changes in effective temperature, luminosity and radius are indeed self-consistent.

- Q11.2

  (a). The mass-loss rate due to nuclear reactions can be determined from the mass equivalent of the solar luminosity:
$$\dot{M} = \frac{L_\odot}{c^2} = 4.27 \times 10^9\,\mathrm{kg\,s^{-1}} = 6.78 \times 10^{-14}\,\mathrm{M_\odot\,yr^{-1}}$$

  (b). The mass-loss rate due to the solar wind is approximately $3 \times 10^{-14}\,\mathrm{M_\odot\,yr^{-1}}$ (*Ostlie & Carroll*, p. 374). This is about half of the mass-loss rate due to nuclear reactions.

  (c). Assuming both mass-loss rates remain constant, the total mass-loss rate of the Sun will be $\sim 10^{-13}\,\mathrm{M_\odot\,yr}$. Over the Sun's main-sequence lifetime of 10^{10} yr, this will give a total mass loss of $10^{-3}\,M_\odot$ — not a significant amount.

- Q11.9 The optical depth τ_λ is related to distance s via the standard equation
$$\tau_\lambda = \int_0^s \kappa_\lambda \rho \,\mathrm{d}s.$$
For uniform opacity and density, this simplifies to
$$\tau_\lambda = \kappa_\lambda \rho s.$$
At wavelength λ_1, the opacity is $\kappa_{\lambda_1} = 0.026\,\mathrm{m^2\,kg^{-1}}$. Assuming a gas density $2.2 \times 10^{-4}\,\mathrm{kg\,m^{-3}}$, the point with optical depth $\tau_\lambda = 2/3$ will be at distance $s_1 = 117\,\mathrm{km}$ into the gas. Likewise, at wavelength λ_2, where the opacity is $\kappa_{\lambda_2} = 0.030\,\mathrm{m^2\,kg^{-1}}$, the point with optical depth $\tau_\lambda = 2/3$ will be at a distance $s_2 = 101\,\mathrm{km}$ into the gas.

  Given that at any wavelength we typically see down to an optical depth $\tau_\lambda = 2/3$, these values demonstrate that we can see further into the gas, by 16 km, at wavelength λ_1 than at wavelength λ_2.

- Q12.12 Hydrostatic equilibrium requires that the inward pull of gravity is balanced by the gradient of the gas pressure,
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g.$$
If the gas pressure is constant, $P = P_0$, then the pressure gradient vanishes on the left-hand side of this equation, and the requirement of hydrostatic equilibrium cannot be satisfied.

  This result indicates that the assumption of constant mass density in an isothermal (constant temperature), constant composition cloud is inconsistent; for, if all of these quantities were constant, then the pressure given by the ideal-gas equation of state,
$$P = \frac{\rho k T}{\mu m_{\rm H}},$$
would be constant also — and hydrostatic equilibrium could not be satisfied.

- Q12.13

(a). The pressure gradient in the cloud can be approximated using a two-point finite difference,

$$\left|\frac{\mathrm{d}P}{\mathrm{d}r}\right| \approx \left|\frac{P_\mathrm{s} - P_\mathrm{c}}{R}\right| \approx \frac{P_c}{R_\mathrm{J}},$$

where the second equality follows from the assumption that the surface pressure P_s is zero. To estimate the central pressure P_c, we can use the ideal gas law

$$P_\mathrm{c} \approx \frac{\rho k T}{\mu m_\mathrm{H}},$$

with ρ and T being the initial cloud density and temperature, respectively. Adopting the values $\rho = 3 \times 10^{-17}\,\mathrm{kg\,m^{-3}}$ and $T = 10\,\mathrm{K}$ for the dense molecular cloud described in Example 12.2.1 of *Ostlie & Carroll*, together with a molecular weight $\mu = 2$, the central pressure is found as $P_\mathrm{c} \approx 1.2 \times 10^{-12}\,\mathrm{N\,m^{-2}}$.

Example 12.2.1 gives the Jeans mass of a typical dense molecular cloud as $M_\mathrm{J} \approx 8\,M_\odot$. For the given density, this translates via eqn. (12.13) to a Jeans radius $R_\mathrm{J} \approx 5 \times 10^{15}\,\mathrm{m} \approx 7.2 \times 10^6\,R_\odot$. Thus, the pressure gradient is estimated as $|\mathrm{d}P/\mathrm{d}r| \approx P_\mathrm{c}/R_\mathrm{J} \approx 2.4 \times 10^{-28}\,\mathrm{N\,m^{-2}\,m^{-1}}$ — not much at all!

(b). We can estimate the strength of the gravity term $GM_r\rho/r^2$ as

$$\frac{GM_r\rho}{r^2} \approx \frac{GM}{R^2}\frac{3M}{4\pi R^3} \approx \frac{3GM^2}{4\pi R^5}.$$

Substituting in the above values for the Jeans mass and radius, this term is found to have a value $1.3 \times 10^{-27}\,\mathrm{N\,m^{-2}\,m^{-1}}$. Comparing this against the pressure gradient found in (a), the latter is around a factor 5 smaller, indicating that pressure forces are relatively unimportant in the dynamics of cloud's collapse; the collapse is essentially free-fall.

(c). The ratio of pressure gradient to gravitational force scales as

$$\frac{\mathrm{d}P}{\mathrm{d}r}\left[\frac{GM_r\rho}{r^2}\right]^{-1} \approx \frac{\rho k T}{\mu m_\mathrm{H} R}\frac{4\pi R^5}{3GM^2}$$

Eliminating the density term on the right-hand side, this becomes

$$\frac{\mathrm{d}P}{\mathrm{d}r}\left[\frac{GM_r\rho}{r^2}\right]^{-1} \approx \frac{3MkT}{4\pi\mu m_\mathrm{H} R^4}\frac{4\pi R^5}{3GM^2},$$

which simplifies to

$$\frac{\mathrm{d}P}{\mathrm{d}r}\left[\frac{GM_r\rho}{r^2}\right]^{-1} \approx \frac{kT}{\mu m_\mathrm{H} G}\frac{R}{M}$$

As the core collapses, R decreases while the other quantities on the right-hand side — in particular, the temperature, since the collapse is isothermal — remain constant. Therefore, the pressure forces continue to decrease relative to $GM_r\rho/r^2$ during the collapse.

- Q12.18

(a). Equation (12.19) of *Ostlie & Carroll* is the equation of motion for gas at radius r in a non-rotating sperical cloud,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM_r}{r^2}.$$

If the cloud is in fact rotating with a (uniform) angular velocity ω, then material in the plane perpendicular to the axis of rotation will experience an outward *centrifugal* acceleration (not centripetal acceleration; the book is wrong in its terminology), and the equation of motion becomes

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM_r}{r^2} + \omega^2 r.$$

Note that this is strictly only valid if the cloud remains spherical as it collapses; any significant departures from sphericity will mean that the gravitational term on the right-hand side no longer scales strictly as r^{-2}.

For the gas at radius r in the equatorial plane, the angular momentum per unit mass is

$$j = \omega r^2.$$

Assuming that this gas does not exchange angular momentum with gas at neighboring radii, this angular momentum does not change as the gas moves inwards, and so ω must change according to the relation

$$\omega r^2 = \omega_0 r_0^2$$

Substituting this into the equation of motion above, we obtain

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM_r}{r^2} + \frac{\omega_0^2 r_0^4}{r^3}.$$

In terms of the radial velocity v_r, this can also be written as

$$v_r \frac{\mathrm{d}v_r}{\mathrm{d}r} = \frac{1}{2}\frac{\mathrm{d}v_r^2}{\mathrm{d}r} = -\frac{GM_r}{r^2} + \frac{\omega_0^2 r_0^4}{r^3}.$$

Integrating with respect to r, we find

$$\frac{1}{2} v_r^2 = \frac{GM_r}{r} - \frac{\omega_0^2 r_0^4}{2r^2} + C.$$

To set the constant of integration C, we note that the cloud is initially at rest, when $r = r_0$. Hence,

$$C = -\frac{GM_r}{r_0} + \frac{\omega_0^2 r_0^4}{2r_0^2},$$

and the final expression for the radial velocity is

$$v_r = \sqrt{2GM_r\left(\frac{1}{r} - \frac{1}{r_0}\right) - \omega_0^2 r_0^4\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)}.$$

To find the final radius where the cloud collapse halts, we look for the point $r = r_\mathrm{f}$ at which $v_r = 0$ in the above equation (other than the $r = r_0$ point we've already imposed via the initial condition). Setting $v_r = 0$ and $r = r_\mathrm{f}$, and further making the assumption that $r_\mathrm{f} \ll r_0$ (so that $1/r_\mathrm{f} \gg 1/r_0$), the equation becomes

$$\frac{2GM_r}{r_\mathrm{f}} - \frac{\omega_0^2 r_0^4}{r_\mathrm{f}^2} = 0$$

Solving this equation gives

$$r_\mathrm{f} = \frac{\omega_0^2 r_0^4}{2GM_r},$$

which is the desired result

(b). Rearranging the above expression, the initial angular velocity can be expressed in terms of the initial mass, initial radius and final mass as

$$\omega_0 = \sqrt{\frac{2GM_r r_\mathrm{f}}{r_0^4}}.$$

Substituting in the supplied values for the mass and initial/final radii, the angular velocity is found as $\omega_0 = 2.65 \times 10^{-16}\,\mathrm{s}^{-1}$.

(c). The initial rotational velocity at the edge of the cloud is $v_{\rm rot,0} = \omega_0 r_0 = 4.08\,{\rm m\,s^{-1}}$ — not much at all!

(d). With complete conservation of angular momentum, we must have

$$\omega_0 I_{\rm sphere,0} = \omega_f I_{\rm disk,f}.$$

In terms of the initial and final radii, this is

$$\frac{2}{5}\omega_0 M_r r_0^2 = \frac{1}{2}\omega_f M_r r_{\rm f}^2$$

Using the value of ω_0 calculated above, plus the supplied values of r_0 and $r_{\rm f}$, the final angular velocity is found as $\omega_f = 2.25\times10^{-10}\,{\rm s^{-1}}$. At a radius $r = 100\,{\rm AU}$, this translates into a rotational velocity $v_{\rm rot} = \omega\, r = 3.37 \times 10^3\,{\rm m\,s^{-1}}$.

(e). The time for one complete revolution is $t = 2\pi/\omega = 2.79 \times 10^{10}\,{\rm s} = 884\,{\rm yr}$. From Kepler's third law (which we can use in its original form because we're dealing with a $1\,M_\odot$ system), the orbital period at 100 AU is $P/({\rm yr}) = ({\rm a/AU})^{3/2} = 1000\,{\rm yr}$. The discrepancy between the two comes from the fact that we have enforced uniform (rigid) rotation in the disk; in reality, the disk would be differentially rotating, with the inner regions rotating with a shorter period than the outer regions.