Homework Assignment 12 — Solutions

- Q10.17

  To confirm that the solution

$$D_0(\xi) = 1 - \frac{\xi^2}{6}, \qquad \xi_1 = \sqrt{6}$$

  is correct, we must check that it satisfies the Lane-Emden equation and the associated boundary conditions. For a polytropic index $n = 0$, the Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right) = -1.$$

  Substituting the solution into the left-hand side of this equation, we have

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}D_0}{\mathrm{d}\xi}\right) = \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}}{\mathrm{d}\xi}\left[1 - \frac{\xi^2}{6}\right]\right)$$

  Working through the differentiations, this becomes

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}D_0}{\mathrm{d}\xi}\right) = \frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\left[-\frac{\xi}{3}\right]\right) = -\frac{1}{\xi^2}\xi^2 = -1.$$

  The right-hand side matches the right-hand side of the Lane-Emden equation, and therefore the solution satisfies the equation.

  To check the boundary conditions, we note that

$$D_0(0) = 1 - \frac{0^2}{6} = 1,$$

$$D_0'(0) = -\frac{0}{3} = 0,$$

  and

$$D_0(\xi_1) = 1 - \frac{\sqrt{6}^2}{6} = 0.$$

  Therefore, all boundary conditions are satisfied, and we have proven that the $n = 0$ polytrope has the given solution.

- Q10.18

  For any polytrope, the density structure is described by

$$\rho(r) = \rho_\mathrm{c}[D_n(r)]^n$$

  (see *Ostlie & Carroll*, p. 336). In the $n = 0$ case, this becomes

$$\rho(r) = \rho_\mathrm{c},$$

  indicating that the density is constant everywhere — that is, the polytrope has a homogeneous density structure.

- Q10.19

  The mass of a polytrope is given by

$$M = -4\pi\lambda_n^3\rho_\mathrm{c}\xi_1^2\left.\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right|_{\xi_1}.$$

  (*Ostlie & Carroll*, p. 338). In the $n = 5$ case, the solution of the Lane-Emden equation is

$$D_5(\xi) = [1 + \xi^2/3]^{-1/2}$$

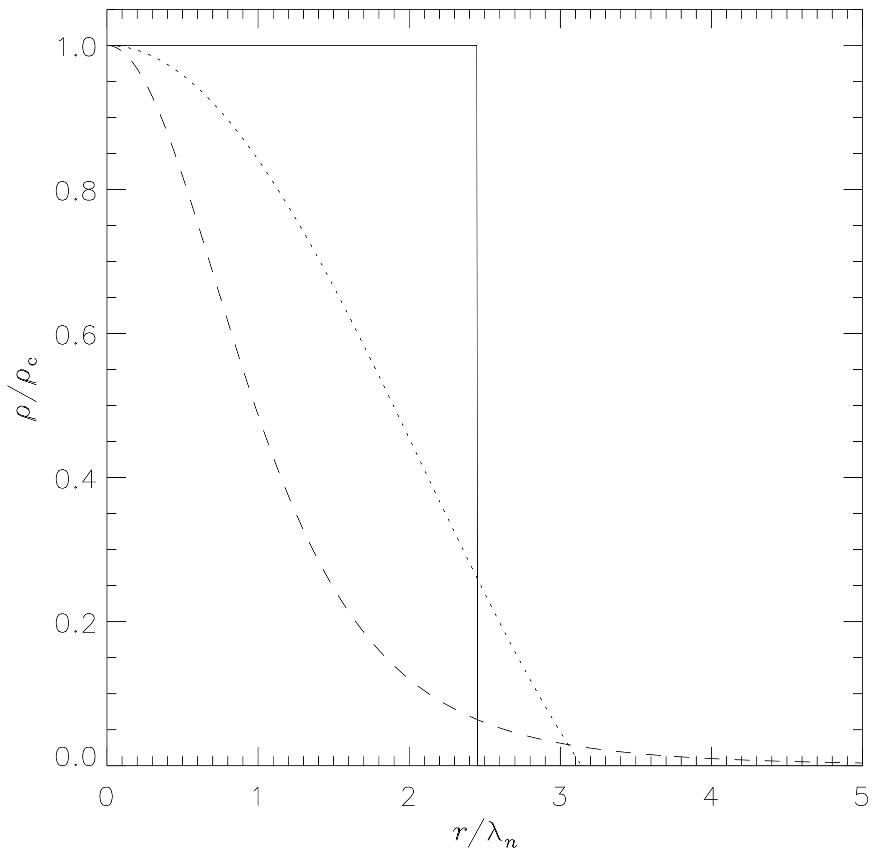

Figure 1: A plot of the normalized density $\rho/\rho_{\rm c}$ as a function of scaled radius r/λ_n, for the $n = 0$ (solid), $n = 1$ (dotted) and $n = 5$ (dashed) polytropes.

(*O & C*, ibid.). The gradient of this solution at ξ_1 is

$$\left.\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right|_{\xi_1} = -\frac{\xi_1}{3(1+\xi_1^2/3)^{3/2}},$$

and thus the mass of the $n = 5$ polytrope is

$$M = 4\pi\lambda_n^3\rho_{\rm c}\frac{\xi_1^3}{3(1+\xi_1^2/3)^{3/2}}.$$

Although $\xi_1 \to \infty$, this mass remains finite because the last term on the right-hand side does not diverge:

$$\lim_{\xi_1\to\infty}\frac{\xi_1^3}{3(1+\xi_1^2/3)^{3/2}} = \sqrt{3},$$

and thus

$$M = \sqrt{3}4\pi\lambda_n^3\rho_{\rm c}$$

- Q10.20

  (a). See Fig. 1 for the plot.

(b). Looking at the plot, it can be concluded that density becomes more concentrated toward the center ($r = 0$) for increasing polytropic index.

(c). An adiabatically convective model has $n = 1.5$ (*Ostlie & Carroll*, pp. 338–339), while a model in radiative equilibrium (i.e., the Eddington standard model) has $n = 3$ (*Ostlie & Carroll*, pp. 339–340). Given the trend noted in (b), the radiative equilibrium model should be more centrally condensed than the adiabatically convective model.

(d). Convection tends to result in shallow temperature gradients, which lead to a 'fluffier', less-condensed star. Radiative diffusion, on the other hand, requires steeper temperature gradients in order to drive the radiative flux through the star; hence, the star must be more centrally condensed.

- Q10.21

The hydrogen-burning lifetime can be estimated as

$$t_H \approx \frac{E_{\rm nuclear}}{L} \approx \frac{0.007 f M c^2}{L}$$

where f is the fraction of the star's mass (assumed to be pure hydrogen) converted from hydrogen into helium; M is the stellar mass; and L the luminosity. The factor 0.007 comes from the fact that the H to He convesion releases 0.7% of the rest mass as energy (see, e.g., Example 10.3.2 of *Ostlie & Carroll*).

For the low-mass star, $f = 1$ (as per the question). With $M = 0.072\,{\rm M}_\odot$ and $\log_{10} L/{\rm L}_\odot = -4.3$, we find $t_H \approx 4.7 \times 10^{21}\,{\rm s} \approx 1.5 \times 10^{14}\,{\rm yr}$ — much longer than the age of the Universe.

For the high-mass star, $f = 0.1$ (from Example 10.3.2). With $M = 85\,{\rm M}_\odot$ and $\log_{10} L/{\rm L}_\odot = 6.006$, we find $t_H \approx 2.7 \times 10^{13}\,{\rm s} \approx 0.87 \times 10^{6}\,{\rm yr}$ — shorter than the duration of human existence on Earth!

- Q10.23

(a). The Eddington luminosity is given by

$$L_{\rm Ed} = \frac{4\pi G c}{\bar{\kappa}} M$$

(*Ostlie & Carroll*, eqn. 10.114). Plugging in the supplied values for the $0.074\,{\rm M}_\odot$ star gives a luminosity $L_{\rm ed} = 9.37 \times 10^4\,{\rm L}_\odot$ — orders of magnitude larger than the star's actual luminosity. Therefore, radiation pressure will not be significant in the stability of low-mass main sequence stars.

(b). For the $120\,{\rm M}_\odot$ star, if the opacity is due to electron scattering, then eqn. (9.27) of *O & C*, with $X = 0.7$, gives $\bar{\kappa} = 0.034\,{\rm m^2\,kg^{-1}}$. Plugging in the other supplied values, the Eddington luminosity is $L_{\rm ed} = 4.59 \times 10^6\,{\rm L}_\odot$. This is somewhat larger than the actual luminosity of the star ($L = 1.79 \times 10^6\,{\rm L}_\odot$), and thus although the star remains below the Eddington limit, it is close to the limit — radiation pressure is important.