Homework Assignment 10 — Solutions

- Q10.1

  The condition for hydrostatic equilibrium (*Ostlie & Carroll*, eqn. 10.6) is

  $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

  Likewise, the Rosseland optical depth is defined in differential form as

  $$\mathrm{d}\tau = -\bar{\kappa}\rho\,\mathrm{d}r$$

  (cf. eqn. 9.15, *ibid*). Combining the two expressions, we have

  $$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{\mathrm{d}P}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{g}{\bar{\kappa}},$$

  which is the desired result.

- Q10.3

  The total chemical energy of the Sun can be expressed as

  $$E = Ne$$

  where N is the total number of particles, and e the chemical energy per particle. Assuming the Sun is pure hydrogen (as per the question), the number of particles is

  $$N = \frac{M_\odot}{m_\mathrm{H}} = 1.19 \times 10^{57}.$$

  Setting $e = 10\,\mathrm{eV} = 1.60 \times 10^{-18}\,\mathrm{J}$, the total chemical energy is $E = 1.90 \times 10^{39}\,J$. With a solar luminosity of $L_\odot = 3.84 \times 10^{26}\,\mathrm{J\,s^{-1}}$, the Sun could therefore last for only

  $$t = \frac{E}{L_\odot} = 4.95 \times 10^{12}\,\mathrm{s} = 1.57 \times 10^5\,\mathrm{yr}$$

  if powered by chemical processes alone. This is clearly much shorter than the age of the Earth (and, indeed, humankind), and therefore it is not possible that the Sun's energy is entirely chemical; there must be some other mechanism powering it.

- Q10.4

  1. To (just) overcome the potential barrier without tunnelling, the proton kinetic energy K must equal the potential energy of the barrier. From p. 301 of *Ostlie & Carroll*, this condition can be expressed as

     $$\frac{1}{2}\mu_p v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

     If the proton has ten times the RMS velocity v_rms, then

     $$v^2 = (10\,v_\mathrm{rms})^2 = 100\,v_\mathrm{rms}^2;$$

     but for the Maxwell-Boltzmann distribution,

     $$\frac{1}{2}m_p v_\mathrm{rms}^2 = \frac{3kT}{2}$$

     where T is the temperature. Combining these expressions,

     $$\frac{1}{2}m_p v^2 = 100\frac{3kT}{2}.$$

Note that the proton mass m_p appears in this last expression, but the reduced mas μ_p is in the first expression. Since $\mu_p = m_p^2/(m_p + m_p) = m_p/2$, we can combine the two expressions to give

$$50\frac{3kT}{2} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}.$$

Solving for the temperature,

$$T = \frac{2}{150k}\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

Using $r = 1\,\mathrm{fm}$ as the nuclear radius, this evaluates to a temperature $T = 2.23 \times 10^8\,\mathrm{K}$.

2. The ratio of particles having a velocity $v = 10v_{\mathrm{rms}}$ to those having $v = v_{\mathrm{rms}}$ is found from eqn. (8.1) as

$$\frac{n_{10}}{n_1} = 100\,\mathrm{e}^{-m_p(100v_{\mathrm{rms}}^2 - v_{\mathrm{rms}}^2)/2kT}$$

With

$$\frac{1}{2}mv_{\mathrm{rms}}^2 = \frac{3kT}{2},$$

this becomes

$$\frac{n_{10}}{n_1} = 100\,\mathrm{e}^{-148.5},$$

which evaluates to a ratio 3.22×10^{-63} — tiny!

3. If the Sun were pure hydrogen, then the total number of hydrogen nucleii (protons) would be

$$N = \frac{M_\odot}{m_p} = 1.19 \times 10^{57}.$$

Suppose also that all of the Sun were at the $T = 2.23 \times 10^8\,\mathrm{K}$ (which certainly isn't the case; the Sun is in fact significantly cooler); then the number of protons having at least ten times the RMS velocity (and therefore able to contribute nuclear energy toward the Sun's luminosity) can be estimated as

$$N_{10} \approx N\frac{n_{10}}{n_1},$$

which evaluates to $\sim 3.8 \times 10^{-6}$. Therefore, not even a single proton would be able to contribute toward the Sun's luminosity!

- Q10.10

From eqn. (10.47) of *Ostlie & Carroll*, the pp energy generation rate can be written as

$$\epsilon_{pp} \approx \rho\, X^2\, f_{pp}\, \psi_{pp}\, C_{pp}\, T_6^4 \;\cdot\; 1.08 \times 10^{-12}\,\mathrm{W\,m^3\,kg^{-2}}.$$

With $T_6 = T/10^6K = 15.696$, $\rho = 1.527 \times 10^5\,\mathrm{kg}$, $X = 0.3397$, $f_{pp} = 1$, and $\psi_{pp} = 1$, and furthermore assuming $C_{pp} = 1$ (this is not given in the question), we find $\epsilon_{pp} \approx 1.2 \times 10^{-3}\,\mathrm{W\,kg^{-1}}$.

From eqn. (10.59) of *O&C*, the CNO energy generation rate can likewise be written as

$$\epsilon_{CNO} \approx \rho\, X\, X_{CNO} T_6^{19.9} \;\cdot\; 8.24 \times 10^{-31}\,\mathrm{W\,m^3\,kg^{-2}}$$

Plugging in the same values for X, ρ and T_6, and with $X_{CNO} = 0.0141$, we obtain $\epsilon_{CNO} \approx 3.8 \times 10^{-4}\,\mathrm{W\,kg^{-1}}$.

The ratio between these values is $\epsilon_{pp}/\epsilon_{CNO} \approx 3$, indicating that the pp chain generates three times more energy at the given T, ρ, etc.

- Q10.12

The Q values are evaluated from the difference $\mathrm{d}m$ between the rest masses of the particles on the left and right-hand sides of the reaction. Note that the masses for ${}^2_1\mathrm{H}$ and ${}^3_2\mathrm{He}$ given in the book are atomic rather than nuclear masses. This isn't a problem *per se*, but it means we should use the atomic mass $1.007825\,u$ for ${}^1_1\mathrm{H}$, rather than the nuclear mass $m_p = 1.007276\,u$.

- ${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \to {}^{2}_{1}\mathrm{H} + e^{+} + \bar{\nu}_e$
  For this reaction, $\mathrm{d}m = 2m({}^{1}_{1}\mathrm{H}) - m({}^{2}_{1}\mathrm{H}) - m(e^{+}) = 0.00100\,u$, and so $Q = (\mathrm{d}m/u)\,931.49\,\mathrm{MeV} = 0.933\,\mathrm{MeV}$.
- ${}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \to {}^{3}_{2}\mathrm{He} + \gamma$
  For this reaction, $\mathrm{d}m = m({}^{2}_{1}\mathrm{H}) + m({}^{1}_{1}\mathrm{H}) - m({}^{3}_{2}\mathrm{He}) = 0.00592\,u$, and so $Q = 5.52\,\mathrm{MeV}$.
- ${}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \to {}^{4}_{2}\mathrm{He} + 2{}^{1}_{1}\mathrm{H}$
  For this reaction, $\mathrm{d}m = 2m({}^{3}_{2}\mathrm{He}) - m({}^{4}_{2}\mathrm{He}) - 2m({}^{1}_{1}\mathrm{H}) = 0.0137\,u$, and so $Q = 12.8\,\mathrm{MeV}$

- Q10.13

  The same approach as in Q10.12 is used:

  1. For this reaction, $\mathrm{d}m = 2m({}^{12}_{6}\mathrm{C}) - m({}^{24}_{12}\mathrm{Mg}) = 0.0150\,u$, and so $Q = (\mathrm{d}m/u)\,931.49\,\mathrm{MeV} = 13.9\,\mathrm{MeV}$. Because Q is positive, this reaction is exothermic.
  2. For this reaction, $\mathrm{d}m = 2m({}^{12}_{6}\mathrm{C}) - m({}^{16}_{8}\mathrm{O}) - 2m({}^{4}_{2}\mathrm{He}) = -0.000116\,u$, and so $Q = -0.108\,\mathrm{MeV}$. Because Q is negative, this reaction is endothermic.
  3. For this reaction, $\mathrm{d}m = m({}^{19}_{9}\mathrm{F}) + m({}^{1}_{1}\mathrm{H}) - m({}^{16}_{8}\mathrm{O}) - m({}^{4}_{2}\mathrm{He}) = 0.00871\,u$, and so $Q = 8.12\,\mathrm{MeV}$. Because Q is positive, this reaction is exothermic.