Homework Assignment 9 — Solutions

- Q9.17

  Within the Eddington approxiation, the specific intensity is given by

$$I = \begin{cases} I_{\mathrm{out}} & 0 \le \theta < \pi/2 \\ I_{\mathrm{in}} & \pi/2 < \theta \le \pi \end{cases}$$

  (see Fig. 9.15 of *Ostlie & Carroll*). From eqn. (9.3) of *O & C*, the mean intensity is defined by

$$\langle I \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} I \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

  Splitting the integral into two intervals (corresponding to the two intervals in the definition above for I), this becomes

$$\langle I \rangle = \frac{1}{4\pi} \left(\int_0^{2\pi} \int_0^{\pi/2} I_{\mathrm{out}} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + \int_0^{2\pi} \int_{\pi/2}^{\pi} I_{\mathrm{in}} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \right).$$

  Doing the integrals[1] (for constant I_{out} and I_{in}),

$$\langle I \rangle = \frac{1}{2} \left(I_{\mathrm{out}} + I_{\mathrm{in}} \right),$$

  which is the correct result (cf. *C & O*, eqn. 9.46).

  A similar procedure is used to find the flux and the radiation pressure. From eqn. (9.8) of *O & C*, the flux is defined by

$$F_{\mathrm{rad}} = \int_0^{2\pi} \int_0^{\pi} I \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

  Splitting the integral into two intervals, and substituting in for I, this becomes

$$F_{\mathrm{rad}} = \int_0^{2\pi} \int_0^{\pi/2} I_{\mathrm{out}} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + \int_0^{2\pi} \int_{\pi/2}^{\pi} I_{\mathrm{in}} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

  Doing the integrals,

$$F_{\mathrm{rad}} = \pi \left(I_{\mathrm{out}} - I_{\mathrm{in}} \right),$$

  which is the correct result (cf. *C & O*, eqn. 9.47).

  Likewise, from eqn. (9.9) of *O & C*, the radiation pressure is defined by

$$P_{\mathrm{rad}} = \frac{1}{c} \int_0^{2\pi} \int_0^{\pi} I \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

  Splitting the integral into two intervals, and substituting in for I, this becomes

$$P_{\mathrm{rad}} = \frac{1}{c} \left(\int_0^{2\pi} \int_0^{\pi/2} I_{\mathrm{out}} \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + \int_0^{2\pi} \int_{\pi/2}^{\pi} I_{\mathrm{in}} \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \right).$$

  Doing the integrals,

$$P_{\mathrm{rad}} = \frac{2\pi}{3c} \left(I_{\mathrm{out}} + I_{\mathrm{in}} \right),$$

  which is the correct result (cf. *C & O*, eqn. 9.48).

[1] The θ integral can be made more straightforward using the subtitution $\mu = \cos\theta$.

- Q9.18

  Solving the radiative transfer equation within the Eddington approximation give the mean intensity as

$$\frac{4\pi}{3}\langle I\rangle = F_{\rm rad}\left(\tau_{\rm v} + \frac{2}{3}\right).$$

  (cf. *Carroll & Ostlie*, eqn. 9.50). Substituting in eqn. (9.46) for the mean intensity, this becomes

$$\frac{2\pi}{3}\left(I_{\rm out} + I_{\rm in}\right) = F_{\rm rad}\left(\tau_{\rm v} + \frac{2}{3}\right).$$

  This equation is combined with eqn. (9.47),

$$F_{\rm rad} = \pi(I_{\rm out} - I_{\rm in}),$$

  to give a pair of simultaneous equation for $I_{\rm out}$ and $I_{\rm in}$. First eliminating $I_{\rm in}$, we obtain

$$\frac{2\pi}{3}\left(I_{\rm out} + I_{\rm out} - \frac{F_{\rm rad}}{\pi}\right) = F_{\rm rad}\left(\tau_{\rm v} + \frac{2}{3}\right).$$

  Rearranging leads to the result

$$I_{\rm out} = \frac{3}{4\pi}F_{\rm rad}\left(\tau_{\rm v} + \frac{4}{3}\right).$$

  Substituting this back into the flux equation (9.47) leads to the corresponding expression for $I_{\rm in}$,

$$I_{\rm in} = \frac{3}{4\pi}F_{\rm rad}\tau_{\rm v}.$$

  (Note that $I = 0$ at $\tau_{\rm v} = 0$, in accordance with the boundary conditions used to derive eqn. 9.50).

  The anisotropy of the radiation field is characterized by the departure of the ratio

$$\frac{I_{\rm out}}{I_{\rm in}} = \frac{\tau_{\rm v} + 4/3}{\tau_{\rm v}}$$

  from unity. For a one percent anisotropy, as asked in the question, we have

$$1.01 = \frac{\tau_{\rm v} + 4/3}{\tau_{\rm v}};$$

  solving for the optical depth, we find $\tau_{\rm v} = 133$.

- Q9.20

  Within the Eddington approximation, eqn. (9.50) of *Ostlie & Carroll* gives an expression for the mean intensity $\langle I\rangle$,

$$\frac{4\pi}{3}\langle I\rangle = F_{\rm rad}\left(\tau_{\rm v} + \frac{2}{3}\right).$$

  For an atmosphere in radiative equilibrium, $\langle I\rangle = S$, and so the source function is given (after a little rearrangement) by

$$S = \frac{3}{4\pi}F_{\rm rad}\left(\tau_{\rm v} + \frac{2}{3}\right).$$

  Applying this expression at optical depth $\tau_{\rm v} = 2/3$ gives $S(\tau_{\rm v} = 2/3) = \pi F_{\rm rad}$, which is the desired result.

- Q9.21

  The general solution of the radiative transfer equation (9.54) is

$$I_\lambda(0) = I_{\lambda,0}\mathrm{e}^{-\tau_{\lambda,0}} + \int_0^{\tau_{\lambda,0}} S_\lambda \mathrm{e}^{-\tau_\lambda}\,\mathrm{d}\tau_\lambda$$

  (I've flipped the integration order relative to *Ostlie & Carroll*, as this is a more intuitive way to write the solution). For a plane-parallel slab with no radiation entering from the outside (as stipulated in the question), $I_{\lambda,0} = 0$ and so

$$I_\lambda(0) = \int_0^{\tau_{\lambda,0}} S_\lambda \mathrm{e}^{-\tau_\lambda}\,\mathrm{d}\tau_\lambda.$$

  If the source function does not depend on position, this simplifies to

$$I_\lambda(0) = S_\lambda \int_0^{\tau_{\lambda,0}} \mathrm{e}^{-\tau_\lambda}\,\mathrm{d}\tau_\lambda = S_\lambda\left(1 - \mathrm{e}^{-\tau_{\lambda,0}}\right).$$

  In the $\tau_{\lambda,0} \gg 1$ limit, the exponential term on the right-hand side goes to zero, and so

$$I_\lambda(0) = S_\lambda.$$

  With the added assumption of thermodynamic equilibrium, the source function S_λ equals the Planck function B_λ, and so

$$I_\lambda(0) = B_\lambda;$$

  that is, the emergent radiation is blackbody radiation, the desired result.

  In the $\tau_{\lambda,0} \ll 1$ limit, the exponential term can be approximated using a first-order Taylor series expansion,

$$\mathrm{e}^{\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0},$$

  and so

$$I_\lambda(0) = S_\lambda\left(1 - 1 + \tau_{\lambda,0}\right) = S_\lambda \tau_{\lambda,0} = S_\lambda \kappa_\lambda \rho L,$$

  where the second equality follows because the opacity κ_λ and density ρ are constant througout the slab. Remembering that the source function is defined as

$$S_\lambda \equiv \frac{j_\lambda}{\kappa_\lambda},$$

  where j_λ and κ_λ are the wavelength-dependent emissivity and opacity of the slab, it follows that

$$I_\lambda(0) = \frac{j_\lambda}{\kappa_\lambda}\kappa_\lambda \rho L = j_\lambda \rho L;$$

  therefore, the slab will show emission lines where j_λ is large, the desired result.

- Q9.22

  The general solution of the radiative transfer equation (9.54) is

$$I_\lambda(0) = I_{\lambda,0}\mathrm{e}^{-\tau_{\lambda,0}} + \int_0^{\tau_{\lambda,0}} S_\lambda \mathrm{e}^{-\tau_\lambda}\,\mathrm{d}\tau_\lambda$$

  If the source function does not depend on position, this simplifies to

$$I_\lambda(0) = I_{\lambda,0}\mathrm{e}^{-\tau_{\lambda,0}} + S_\lambda \int_0^{\tau_{\lambda,0}} \mathrm{e}^{-\tau_\lambda}\,\mathrm{d}\tau_\lambda = I_{\lambda,0}\mathrm{e}^{-\tau_{\lambda,0}} + S_\lambda\left(1 - \mathrm{e}^{-\tau_{\lambda,0}}\right).$$

  In the $\tau_{\lambda,0} \gg 1$ limit, the exponential terms on the right-hand side go to zero, and so

$$I_\lambda(0) = S_\lambda.$$

With the added assumption of thermodynamic equilibrium, the source function S_λ equals the Planck function B_λ, and so

$$I_\lambda(0) = B_\lambda;$$

that is, the emergent radiation is blackbody radiation, the desired result.

In the $\tau_{\lambda,0} \ll 1$ limit, the exponential term can be approximated using a first-order Taylor series expansion,

$$\mathrm{e}^{\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0},$$

and so

$$I_\lambda(0) = I_{\lambda,0}(1 - \tau_{\lambda,0}) + S_\lambda\,(1 - 1 + \tau_{\lambda,0}) = I_{\lambda,0}(1 - \tau_{\lambda,0}) + S_\lambda \tau_{\lambda,0}.$$

Since we have assumed $\tau_{\lambda,0} < 1$, this value lies somewhere between $I_{\lambda,0}$ and S_λ. To obtain absorption lines superimposed on the spectrum of the incident radiation ($I_{\lambda,0}$), we require $I_\lambda(0) < I_{\lambda,0}$, which means that $S_\lambda < I_{\lambda,0}$. Conversely, to obtain emission lines superimposed on the spectrum of the incident radiation, we require $I_\lambda(0) > I_{\lambda,0}$, which means that $S_\lambda > I_{\lambda,0}$. These are the desired results.

- Q9.23

The general equation for the emergent intensity from a plane-parallel atmosphere is

$$I_\lambda(0) = \int_0^\infty S_\lambda \sec\theta\, \mathrm{e}^{\tau_{\lambda,\mathrm{v}} \sec\theta}\, \mathrm{d}\tau_{\lambda,\mathrm{v}}$$

(from eqn. 9.55 of *Ostlie & Carroll*). Assuming a linearly varying source function

$$S_\lambda = a_\lambda + b_\lambda \tau_{\lambda,\mathrm{v}},$$

the intensity becomes

$$I_\lambda(0) = \int_0^\infty (a_\lambda + b_\lambda \tau_{\lambda,\mathrm{v}}) \sec\theta\, \mathrm{e}^{\tau_{\lambda,\mathrm{v}} \sec\theta}\, \mathrm{d}\tau_{\lambda,\mathrm{v}}$$

This integrates to

$$I_\lambda(0) = \left[-\mathrm{e}^{-\sec\theta \tau_{\lambda,\mathrm{v}}} (a_\lambda + b_\lambda \cos\theta + b_\lambda \tau_{\lambda,\mathrm{v}})\right]_0^\infty = a_\lambda + b_\lambda \cos\theta,$$

which is the desired result.