Homework Assignment 8 — Solutions

- Q9.11

  (a). The mean free path is calculated from the opacity and density as

  $$\ell = \frac{1}{\kappa\rho}$$

  (cf. *Ostlie & Carroll*, above eqn. 9.15). With $\kappa = \bar{\kappa} = 0.217\,\mathrm{m^2\,kg^{-1}}$ and $\rho = 1.53 \times 10^5\,\mathrm{kg\,m^{-3}}$, the mean free path is found as $\ell = 3.01 \times 10^{-5}\,\mathrm{m}$

  (b). From eqn. (9.30) of *O & C*, the net ('as-the-crow-flies') distance d moved by a photon that random walks through N steps (of average length equal to the mean free path ℓ) is

  $$d = \ell\sqrt{N}.$$

  For a photon walking from the center of Sun to the surface, with constant ℓ and assuming no destruction and recreation, the number of steps taken can be found by setting $d = R_\odot$ and solving for N,

  $$N = \frac{R_\odot^2}{\ell^2}.$$

  The total path length traveled by the photon is then given by

  $$D = N\ell = \frac{R_\odot^2}{\ell}.$$

  Since the photon travels at the speed of light, the time taken for it to traverse this path is

  $$t = \frac{D}{c} = \frac{R_\odot^2}{\ell c}.$$

  For $\ell = 3.01 \times 10^{-5}\,\mathrm{m}$ from above, this expression gives an average escape time of $t = 5.36 \times 10^{13}\,\mathrm{s} = 1.70 \times 10^6\,\mathrm{yr}$.

- *Repeat the second part of problem 9.11, but assume a typical photon mean free path of 0.003 m instead. How does this estimate of the leakage time for radiative diffusion compare with the time for a photon to escape if it doesn't interact at all with matter and just flies straight through the Sun?*

  With $\ell = 0.003\,\mathrm{m}$, the leakage time is $t = 5.39 \times 10^{11}\,\mathrm{s} = 1.71 \times 10^4\,\mathrm{yr}$. If the photon didn't interact at all with matter (i.e., behaved like a neutrino), then the escape time would be $t = R_\odot/c = 2.32\,\mathrm{s}$ — a factor of $R_\odot/\ell = 2.32 \times 10^{11}$ quicker!

- *We can form a crude estimate of the luminosity of the Sun as follows: we approximate the total energy in radiation inside the Sun as the volume of the Sun times the radiation energy density $u = aT^4$, using a typical interior temperature of $4.5 \times 10^6\,\mathrm{K}$. We then estimate the Sun's luminosity as this radiation energy over the leakage time from the previous (supplemental) problem. How does this estimated luminosity compare with the listed value $L_\odot = 3.839 \times 10^{26}\,\mathrm{W}$?*

  Following this procedure, the radiation energy will be given by

  $$E = \frac{4\pi R_\odot^3}{3} aT^4,$$

  which, with the given temperature, evaluates to $E = 4.38 \times 10^{38}\,\mathrm{J}$. Dividing by the leakage timescale of $5.39 \times 10^{11}\,\mathrm{s}$ gives a luminosity estimate of $L_\odot = 8.13 \times 10^{26}\,\mathrm{W}$, which is only factor ~ 2 larger than the listed value.

- *We can calculate the (thermal) plasma energy density at the center of the Sun as $u_{\mathrm{plasma},c} = \frac{3}{2} n_c k T_c$, with $n_c \sim 10^{32}\,\mathrm{cm}^{-3}$ and $T_c = 1.5 \times 10^7\,\mathrm{K}$. How does the plasma energy density compare with the radiation energy density ($u_{\mathrm{rad},c} = a T_c^4$) at the center of Sun? If we assume the same ratio of plasma energy to radiation energy throughout the Sun's interior, and use our estimate of the leakage time from the first supplemental problem, about how long would it take for the Sun to "turn off" if all the nuclear reactions in the Sun's core stopped?*

  Using the expressions given, the plasma energy density is $u_{\mathrm{plasma},c} = 3.11 \times 10^{16}\,\mathrm{J\,cm^{-3}} = 3.11 \times 10^{22}\,\mathrm{J\,m^{-3}}$, while the radiation energy density is $u_{\mathrm{rad},c} = 3.83 \times 10^{13}\,\mathrm{J\,m^{-3}}$ — around 9 orders of magnitude smaller. An estimate of the rate of loss of energy (i.e., the Sun's luminosity) can be found by applying the second supplemental problem 'in reverse':

  $$L_\odot \approx \frac{V_\odot u_{\mathrm{rad},c}}{t_{\mathrm{leak}}},$$

  where $V_\odot \equiv 4\pi R_\odot^3/3$ is the volume of the Sun, and t_{leak} is the leakage time. Without nuclear reactions in the core, the Sun will exhaust its reserves of thermal energy (heat) over a turn-off timescale

  $$t_{\mathrm{off}} \approx \frac{V_\odot u_{\mathrm{plasma},c}}{L_\odot}.$$

  Combining the two expressions,

  $$t_{\mathrm{off}} \approx \frac{u_{\mathrm{plasma},c}}{u_{\mathrm{rad},c}} t_{\mathrm{leak}}.$$

  Plugging in the supplied energy density values, and $t_{\mathrm{leak}} = 5.39 \times 10^{11}\,\mathrm{s}$ from the first supplemental problem, gives $t_{\mathrm{leak}} \approx 4.4 \times 10^{20}\,\mathrm{s} \approx 1.4 \times 10^{13}\,\mathrm{yr}$ — much longer than the age of the Universe!

  It turns out there was a typo in the original question; the supplied number density was given in m^{-3}, not cm^{-3} as stated. With this error fixed, the leakage time is $t_{\mathrm{leak}} \approx 1.4 \times 10^7\,\mathrm{yr}$, which is far more sensible.

- Q9.12

  From Kirchhoff's laws (Sec. 5.1 of *Ostlie & Carroll*), a hot gas in front of a cooler background radiation source (as is the case for an outward-*increasing* temperature) would show emission lines at wavelengths where the opacity is large.

- Q9.13

  The shell of gas would appear as a ring if it were *optically thin*. This is because in the optically thin limit, we see emission from all parts of the shell (and not just from the very outer layers). Sight lines at the limb of the shell pass through a greater amount of gas than those coming from the center (due to geometrical projection), and therefore appear brighter. Essentially, this is the reverse of limb darkening!

- Q9.15

  To derive eqn. (9.35), we start with the radiative transfer equation given in (9.34) of *Ostlie & Carroll*,

  $$-\frac{1}{\kappa_\lambda \rho} \frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = I_\lambda - S_\lambda.$$

  Rearranging,

  $$\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} + \kappa_\lambda \rho I_\lambda = \kappa_\lambda \rho S_\lambda.$$

  This is a first-order inhomogeneous differential equation, which — for constant κ_λ — is solved by multiplying through by the integrating factor $\mathrm{e}^{\kappa_\lambda \rho s}$. This gives

  $$\mathrm{e}^{\kappa_\lambda \rho s} \frac{\mathrm{d}I_\lambda}{\mathrm{d}s} + \kappa_\lambda \rho\, \mathrm{e}^{\kappa_\lambda \rho s} I_\lambda = \kappa_\lambda \rho\, \mathrm{e}^{\kappa_\lambda \rho s} S_\lambda,$$

which can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{e}^{\kappa_\lambda \rho s} I_\lambda\right) = \kappa_\lambda \rho\, \mathrm{e}^{\kappa_\lambda \rho s} S_\lambda.$$

Integrating with respect to s, with a constant source function S_λ, leads to

$$\mathrm{e}^{\kappa_\lambda \rho s} I_\lambda = \mathrm{e}^{\kappa_\lambda \rho s} S_\lambda + C,$$

where C is a constant of integration. If $I_\lambda = I_{\lambda,0}$ at $s = 0$, this constant must be chosen as

$$C = I_{\lambda,0} - S_\lambda,$$

giving

$$I_\lambda(s) = S_\lambda + (I_{\lambda,0} - S_\lambda)\mathrm{e}^{-\kappa_\lambda \rho s} = I_{\lambda,0}\mathrm{e}^{-\kappa_\lambda \rho s} + S_\lambda\left(1 - \mathrm{e}^{-\kappa_\lambda \rho s}\right).$$

This is the desired result.

- Q9.16

(a). The starting point is the ratiative transfer equation (RTE) given in (9.34) of *Ostlie & Carroll*,

$$-\frac{1}{\kappa_\lambda \rho}\frac{\mathrm{d}I_\lambda}{\mathrm{d}s} = I_\lambda - S_\lambda.$$

The distance $\mathrm{d}s$ along a ray at an angle θ' to the radial direction can be related to the change in radius $\mathrm{d}r$ via

$$\mathrm{d}s = \mathrm{d}r \sec\theta'$$

(see, e.g., Fig. 9.16 of *O & C*). Combining the two expressions,

$$-\frac{\cos\theta'}{\kappa_\lambda \rho}\frac{\mathrm{d}I_\lambda}{\mathrm{d}r} = I_\lambda - S_\lambda,$$

which is the desired result.

(b). Multiplying the above equation by $\cos\theta'$ and integrating over all solid angles gives

$$-\frac{1}{\kappa_\lambda \rho}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{2\pi}\int_0^{\pi} I_\lambda \cos^2\theta' \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi} I_\lambda \cos^2\theta' \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\phi$$

(note that the S_λ term on the right-hand side vanishes, because it is independent of direction). Using the definition of specific radiative flux (eqn. 9.8 of *O & C*) and radiation pressure (eqn. 9.9), this can be written as

$$-\frac{c}{\kappa_\lambda \rho}\frac{\mathrm{d}P_{\mathrm{rad},\lambda}}{\mathrm{d}r} = F_{\mathrm{rad},\lambda}.$$

Integrating over all wavelengths, this becomes

$$-\frac{c}{\rho}\int_0^\infty \frac{1}{\kappa_\lambda}\frac{\mathrm{d}P_{\mathrm{rad},\lambda}}{\mathrm{d}r}\,\mathrm{d}\lambda = \int_0^\infty F_{\mathrm{rad},\lambda}\,\mathrm{d}\lambda \equiv F_{\mathrm{rad}}.$$

In general, the integral on the left-hand side cannot be simplified. *However*, if we are beneath the outermost layers, the radiation field will be close to blackbody, and we can approximate the radiation pressure using the Planck function,

$$P_{\mathrm{rad},\lambda} = \frac{4\pi}{3c}B_\lambda$$

(see eqn. 9.10 of *O & C*). The gradient of the radiation pressure can then be evaluated as

$$\frac{\mathrm{d}P_{\mathrm{rad},\lambda}}{\mathrm{d}r} = \frac{4\pi}{3c}\frac{\mathrm{d}B_\lambda}{\mathrm{d}r} = \frac{4\pi}{3c}\frac{\partial B_\lambda}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}r},$$

where the second equality holds because B_λ depends on the temperature T but not on the density. Substituting this expression back into the RTE gives

$$-\frac{4\pi}{3\rho}\frac{\mathrm{d}T}{\mathrm{d}r}\int_0^\infty \frac{1}{\kappa_\lambda}\frac{\partial B_\lambda}{\partial T}\,\mathrm{d}\lambda = F_{\mathrm{rad}}.$$

Making use of the definition of the Rosseland mean opacity (eqn. 9.21 of *O & C*), the integral becomes

$$-\frac{4\pi}{3\rho}\frac{\mathrm{d}T}{\mathrm{d}r}\frac{1}{\bar{\kappa}}\int_0^\infty \frac{\partial B_\lambda}{\partial T}\,\mathrm{d}\lambda = F_{\mathrm{rad}}.$$

With further application of the chain rule,

$$-\frac{4\pi}{3\rho}\frac{1}{\bar{\kappa}}\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^\infty B_\lambda\,\mathrm{d}\lambda\right) = F_{\mathrm{rad}}.$$

But since

$$P_{\mathrm{rad}} = \frac{4\pi}{3c}\int_0^\infty B_\lambda\,\mathrm{d}\lambda$$

(eqn. 9.11 of *O & C*), we have

$$-\frac{c}{\rho\bar{\kappa}}\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = F_{\mathrm{rad}}.$$

A little rearrangement gives

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = \frac{\bar{\kappa}\rho}{c}F_{\mathrm{rad}},$$

which is the desired result.