Homework Assignment 7 — Solutions

- Q9.1

  The total energy of blackbody photons in an eye of volume $V_{\rm eye}$ is given by

$$E_{\rm bb} = V_{\rm eye} u = V_{\rm eye} a T^4$$

  where the second equality comes from the expression $u = aT^4$ for the energy density of blackbody radiation. With $V_{\rm eye} = 4\pi r^3/3 = 1.41 \times 10^{-5}\,{\rm m}^3$ and $T = 37^\circ\,{\rm C} = 310\,{\rm K}$, the energy of blackbody photons follows as $E_{\rm bb} = 9.88 \times 10^{-11}\,{\rm J}$.

  To consider the contribution of the light bulb toward the radiation in the eye, the net amount of energy entering the pupil in time $\mathrm{d}t$ is

$$\mathcal{E} = \frac{a_{\rm pup}}{4\pi d^2} L\,\mathrm{d}t,$$

  where L is the bulb's luminosity, $a_{\rm pup}$ is the pupil area, and d the distance betwen the bulb and the pupil. This energy has a density

$$u = \frac{\mathcal{E}}{a_{\rm pup} c \mathrm{d}t} = \frac{L}{4\pi d^2 c}$$

  (from arguments similar to those on p. 233 of *Ostlie & Carroll*). Within the eye, assuming no beam divergence, the energy from the bulb occupies a cylindrical volume extending from the pupil to the retina (length $2r$, area $a_{\rm pup}$), and so the total energy is

$$E_{\rm bulb} = 2r a_{\rm pup} u = \frac{2r a_{\rm pup} L}{4\pi d^2 c}$$

  Plugging in the values of $L = 100,{\rm W}$, $d = 1\,{\rm m}$, and the dimensions of the eye and pupil, the 'bulb' energy within the eye is found as $E_{\rm bulb} = 7.96 \times 10^{-15}\,{\rm J}$

  This value is, surprisingly, much smaller than the energy of the blackbody photons. However, the receptors in our eyes are only sensitive to light in the visible range; whereas the blackbody photons, with a Wien-peak wavelength of $\lambda_{\max} = 0.3\,{\rm cm}/T \approx 10\,\mu{\rm m}$, are in the mid-infrared. Thus, it appears dark when we close our eyes because we cannot detect these blackbody photons.

- Q9.2

  (a). The number density of blackbody photons with wavelength between λ and $\lambda + \mathrm{d}\lambda$ is given by dividing the blackbody energy $u_\lambda\,\mathrm{d}\lambda$ in the wavelength range by the photon energy $h\nu = hc/\lambda$:

$$n_\lambda\,\mathrm{d}\lambda = \frac{u_\lambda}{hc/\lambda}\,\mathrm{d}\lambda = \frac{\lambda u_\lambda}{hc}\,\mathrm{d}\lambda.$$

  Using the expression for the blackbody energy density u_λ (cf. *Ostlie & Carroll*, eqn. 9.5), this becomes

$$n_\lambda\,\mathrm{d}\lambda = \frac{8\pi/\lambda^4}{\exp(hc/\lambda kT) - 1}\,\mathrm{d}\lambda.$$

  (b). The total number density of photons can be obtained by integrating $n_\lambda d\lambda$ over all wavelengths,

$$n = \int_0^\infty n_\lambda\,\mathrm{d}\lambda = \int_0^\infty \frac{8\pi/\lambda^4}{\exp(hc/\lambda kT) - 1}\,\mathrm{d}\lambda.$$

  This is a messy integral, but can be evaluated using software such as *Mathematica* to find the result

$$n = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3$$

  where ζ is the Riemann zeta function, which has $\zeta(3) \approx 1.202$.

  For an oven at $477\,{\rm K}$, this expression gives a number density $n = 2.2 \times 10^{15}\,{\rm m}^{-3}$; with a volume of $0.5\,{\rm m}^3$, the total number of photons is therefore 1.1×10^{15}.

- Q9.3

  (a). The average energy per photon is given by the ratio of the total energy density

  $$u = aT^4$$

  and the total number density of photons, which from Q9.2(b) is

  $$n = 16\pi\zeta(3)\left(\frac{kT}{hc}\right)^3.$$

  Thus,

  $$\frac{u}{n} = \frac{ah^3c^3}{16\pi\zeta(3)k^4}kT = 2.70\,kT,$$

  which is the desired result

  (b). For the center of the Sun $T = 1.57 \times 10^7\,\mathrm{K}$, giving $u/n = 5.85 \times 10^{-16}\,\mathrm{J} = 3.65\,\mathrm{keV}$; whereas for the photosphere $T = 5777\,\mathrm{K}$, giving $u/n = 2.15 \times 10^{-19}\,\mathrm{J} = 1.34\,\mathrm{eV}$.

- Q9.4

  The total radiation pressure is given by

  $$P_{\mathrm{rad}} = \int_0^\infty P_{\mathrm{rad},\lambda}\,\mathrm{d}\lambda.$$

  Since blackbody radiation is isotropic, eqn. (9.10) of *Ostlie & Carroll* can be used to find $P_{\mathrm{rad},\lambda}$ as

  $$P_{\mathrm{rad},\lambda} = \frac{4\pi}{3c}I_\lambda = \frac{4\pi}{3c}B_\lambda,$$

  where B_λ is the usual Planck function. Combining these equations,

  $$P_{\mathrm{rad}} = \frac{4\pi}{3c}\int_0^\infty B_\lambda\,\mathrm{d}\lambda.$$

  But as demonstrated in eqn. (9.7), the integral over the Planck function is related to the total energy density u,

  $$\int_0^\infty B_\lambda\,\mathrm{d}\lambda = \frac{c}{4\pi}u,$$

  Hence,

  $$P_{\mathrm{rad}} = \frac{4\pi}{3c}\frac{c}{4\pi}T^4 = \frac{1}{3}u,$$

  which is the desired result.

- Q9.5

  From eqn (9.8) of *Ostlie & Carroll*, the specific radiative flux produced by a blackbody surface with $I_\lambda = B_\lambda$ is

  $$F_{\mathrm{rad},\lambda} = \int_0^{2\pi}\int_0^{\pi/2} B_\lambda \cos\theta \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

  (note that the θ integral only extends to $\pi/2$, since we assume only *outward* radiation). Because B_λ does not depend on direction, we have

  $$F_{\mathrm{rad},\lambda} = B_\lambda \int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \pi B_\lambda.$$

  To calculate the bolometric radiative flux, we next integrate $F_{\mathrm{rad},\lambda}$ over all wavelengths:

  $$F_{\mathrm{rad}} = \int_0^\infty F_{\mathrm{rad},\lambda}\,\mathrm{d}\lambda = \pi\int_0^\infty B_\lambda\,\mathrm{d}\lambda.$$

From eqn. (9.7), the integral over the Planck function is

$$\int_0^\infty B_\lambda \, \mathrm{d}\lambda = \frac{\sigma}{\pi} T^4,$$

whence we obtain

$$F_{\mathrm{rad}} = \pi \frac{\sigma}{\pi} T^4 = \sigma T^4.$$

As a final step, we multiply this flux (= energy loss per unit surface area) by the total surface area ($= 4\pi R^2$), to obtain the bolometric luminosity of a blackbody star as

$$L = 4\pi R^2 \sigma T^4.$$

As desired, this result is in agreement with eqn. (3.17) (allowing for the fact that $T = T_{\mathrm{e}}$ for the blackbody star).

- Q9.6

  The mean free path is given by eqn (9.12) of *Ostlie & Carroll*,

$$\ell = \frac{1}{n\sigma}$$

  where n is total number density of particles, and σ is the collision cross section. The number density is related to the mass density by

$$n = \frac{\rho}{\mu m_{\mathrm{H}}},$$

  where μ is the mean molecular weight, expressed in units of the hydrogen atom mass $m_{\mathrm{H}} = 1.67 \times 10^{-27}\,\mathrm{kg}$. Assuming that the air is all nitrogen (rather than the actual $\sim 80\%$), and taking $\rho = 1.2\,\mathrm{kg\,m^{-3}}$ and $\mu \approx 28$, the number density is found as $n = 2.57 \times 10^{25}\,\mathrm{m^{-3}}$. Combining this with the nitrogen interaction cross section[1] $\sigma = \pi(2r)^2 = 1.26 \times 10^{-19}\,\mathrm{m^2}$ (for $r = 0.1\,\mathrm{nm}$), the mean free path is finally obtained as $\ell = 3.10 \times 10^{-7}\,\mathrm{m}$.

  If the nitrogen molecules are travelling at the RMS speed, then the time between collisions is

$$\Delta t = \frac{\ell}{v_{\mathrm{rms}}}.$$

  Assuming a Maxwell-Boltzmann velocity distribution, eqn. (8.3) gives

$$v_{\mathrm{rms}} = \sqrt{\frac{3kT}{\mu m_{\mathrm{H}}}},$$

  which, with $T = 300\,\mathrm{K}$ and $\mu \approx 28$, gives $v_{\mathrm{rms}} = 515\,\mathrm{m s^{-1}}$. Hence, the time between collisions is found as $\Delta t = 6.02 \times 10^{-10}\,\mathrm{s}$.

- Q9.7

  Suppose we can see some fixed optical depth τ through the Earth's atmosphere. This optical depth translates into a physical distance $\mathrm{d}z$ via

$$\mathrm{d}z = \frac{\tau}{\kappa\rho},$$

  where the opacity κ and density ρ are assumed constant. Taking $\kappa = 0.03\,\mathrm{m^2\,kg^{-1}}$ from Example 9.2.2, and $\rho = 1.2\,\mathrm{kg\,m^{-3}}$, gives

$$\mathrm{d}z = \tau \times 27.8\,\mathrm{m}.$$

  In reality we don't see photons all coming from the same optical distance away — but on average[2], the photons reaching us will have traveled through an optical depth $\tau = 1$. Hence, the distance seen is $\mathrm{d}z = 27.8\,\mathrm{m}$ — much smaller than how far we really see on Earth (assuming there's no fog!).

[1] The discussion on p. 240 of *Ostlie & Carroll* explains where the factor 2 comes from.

[2] This can be demonstrated by considering that there is a $\exp(-\tau)$ probability that a photon originating an optical depth τ away will reach an observer. If photons are emitted at all optical depths with equal probabilty, then the average optical depth traveled by photons reaching the eye is $\langle\tau\rangle = \int_0^\infty \tau \exp(-\tau)\,\mathrm{d}\tau = 1$