Homework Assignment 6 — Solutions

- Q8.1

  Room temperature is 294 K, so $kT = 4.06 \times 10^{-14}\,\mathrm{J} = 0.0253\,\mathrm{eV} \approx 1/40\,\mathrm{eV}$. $kT = 1\,\mathrm{eV}$ when $T = 11,600\,\mathrm{K}$, and $kT = 13.6\,\mathrm{eV}$ when $T = 158,000\,\mathrm{K}$.

- Q8.5

  The Boltzmann equation gives us

  $$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 4\,e^{-10.2\,\mathrm{eV}/kT},$$

  so we have $\frac{n_2}{n_1} = 1\%$ when $e^{-10.2\,\mathrm{eV}/kT} = 0.0025$. Taking the logarithm of both sides gives $-10.2\,\mathrm{eV}/kT = -5.99$, and solving for T gives $T = 1.97 \times 10^4\,\mathrm{K}$. The same procedure for $\frac{n_2}{n_1} = 10\%$ gives $T = 3.21 \times 10^4\,\mathrm{K}$.

- Q8.6

  (a). From the Boltzmann equation,

  $$\frac{n_3}{n_1} = \frac{g_3}{g_1} e^{-(E_3-E_1)/kT} = 9\,e^{-12.1\,\mathrm{eV}/kT},$$

  which gives $\frac{n_3}{n_1} = 1$ when $T = 6.38 \times 10^4\,\mathrm{K}$.

  (b). When $T = 85,400\,\mathrm{K}$, $\frac{N_3}{N_1} = 9\,e^{-1.64} = 1.74$, so if there are $N_1 = N$ atoms in the $n = 1$ state then there are $N_3 = 1.74\,N$ atoms in the second excited state ($n = 3$).

  (c). As $T \to \infty$, the exponential factor in the Boltzmann equation becomes 1 for every value of n, so the Boltzmann equation predicts that the distribution of electrons mirrors the values of the degeneracies: level n has a number of electrons proportional to n^2.

  In reality, as $T \to \infty$ all atoms ionize, and all the electrons are free electrons.

- Q8.7

  We have $Z_{\mathrm{I}} = g_1 + g_2 e^{-10.2\,\mathrm{eV}/kT} + g_3 e^{-12.1\,\mathrm{eV}/kT}$, with $g_1 = 2(1)^2 = 2$, $g_2 = 2(2)^2 = 8$, and $g_3 = 2(3)^2 = 18$. At $T = 10,000\,\mathrm{K}$, this gives $Z_{\mathrm{I}} = 2 + 5.81 \times 10^{-5} + 1.46 \times 10^{-5}$. We can see that the second and third terms are much smaller than $g_1 = 2$, so we have $Z_{\mathrm{I}} \approx 2$.

- Q8.9

  (a). The Saha equation is

  $$\frac{N_{\mathrm{II}}}{N_{\mathrm{I}}} = \frac{2Z_{\mathrm{II}}}{n_e Z_{\mathrm{I}}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT}.$$

  We also have charge conservation

  $$n_e V = N_{\mathrm{II}}$$

  and conservation of nucleons

  $$N_t = \rho V/(m_p + m_e) \approx \rho V/m_p,$$

  where $N_t = N_{\mathrm{I}} + N_{\mathrm{II}}$ is the total number of hydrogen nuclei (whether in neutral hydrogen or ionized hydrogen).

  The first step is to replace the partition function Z_{I} with the ground state degeneracy $g_1 = 2$—recall from question 8.7 that this is a good approximation at $10,000\,\mathrm{K}$, and in fact it is a good approximation for $kT \ll 10.2\,\mathrm{eV}$, or $T \ll 120,000\,\mathrm{K}$. The partition function Z_{II} of the proton is 1. We can now write the Saha equation (for this pure hydrogen case, and assuming $T \ll 120,000\,\mathrm{K}$) as

$$\frac{N_{\mathrm{II}}}{N_{\mathrm{I}}} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT} = \frac{V}{N_{\mathrm{II}}}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT},$$

where we use charge conservation to get rid of n_e in the final expression. We also have $N_t = N_{\mathrm{I}} + N_{\mathrm{II}} = N_{\mathrm{I}}\left(1 + N_{\mathrm{II}}/N_{\mathrm{I}}\right)$; from the Saha equation, we get $N_t = N_{\mathrm{I}}\left(1 + x/N_{\mathrm{II}}\right)$, where $x = V\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT}$. We can rearrange this as $N_{\mathrm{I}} = N_t/\left(1 + x/N_{\mathrm{II}}\right)$. Putting this in the Saha equation gives

$$\frac{N_{\mathrm{II}}}{N_t/\left(1 + x/N_{\mathrm{II}}\right)} = \frac{x}{N_{\mathrm{II}}},$$

or

$$\frac{N_{\mathrm{II}}^2}{N_t}\left(1 + x/N_{\mathrm{II}}\right) = x.$$

This gives

$$\frac{N_{\mathrm{II}}^2}{N_t} + x\frac{N_{\mathrm{II}}}{N_t} - x = 0.$$

Dividing through by N_t, we have

$$\left(\frac{N_{\mathrm{II}}}{N_t}\right)^2 + \frac{x}{N_t}\left(\frac{N_{\mathrm{II}}}{N_t}\right) - \frac{x}{N_t} = 0. \tag{1}$$

Now we use the conservation of nucleons, $N_t \approx \rho V/m_p$ and our expression for x, $x = V\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT}$, to get

$$\frac{x}{N_t} = \frac{V\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT}}{\rho V/m_p} = \left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT},$$

which we plug in to equation 1 to get

$$\left(\frac{N_{\mathrm{II}}}{N_t}\right)^2 + \left(\frac{N_{\mathrm{II}}}{N_t}\right)\left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT} - \left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{I}}/kT} = 0.$$

(b). Taking $\rho = 10^{-6}\,\mathrm{kg\,m^{-3}}$, we have

$$\left(\frac{N_{\mathrm{II}}}{N_t}\right)^2 + 4.04\,(T/1\,\mathrm{K})^{3/2}\,e^{-158,000\,\mathrm{K}/T}\left(\frac{N_{\mathrm{II}}}{N_t}\right) - 4.04\,(T/1\,\mathrm{K})^{3/2}\,e^{-158,000\,\mathrm{K}/T} = 0.$$

Defining $b \equiv 4.04\,(T/1\,\mathrm{K})^{3/2}\,e^{-158,000\,\mathrm{K}/T}$, we have the quadratic equation $y^2 + by - b = 0$, where y is the ionization fraction N_{II}/N_t. When $b \gg 1$, the quadratic term becomes negligible (remember, y must be between 0 and 1, so y^2 is also between 0 and 1), and the equation reduces to $by - b \approx 0$, which has solution $y \approx 1$. When $b \ll 1$, the linear and constant terms become negligible and we have $y^2 \approx 0$, so $y \approx 0$. Plugging in $T = 5000\,\mathrm{K}$ gives $b = 2.7 \times 10^{-8} \ll 1$, and plugging in $T = 25,000\,\mathrm{K}$ gives $b = 2.9 \times 10^4 \gg 1$, so we expect that our graph will go from $y = N_{\mathrm{II}}/N_t \approx 0$ at the low temperature end to $N_{\mathrm{II}}/N_t \approx 1$ at the high temperature end, with a transition at temperatures where $b \approx 1$ (in fact, $N_{\mathrm{II}}/N_t = 0.5$ when $b = 0.5$). Looking at the plot, we see that these predictions are correct—the transition from fully neutral to fully ionized happens at a temperature of about 9900 K, at which point $b \approx 0.5$.

- Q8.13

  From Example 8.1.5, we have $T = 5777\,\mathrm{K}$ and $P_e = 1.5\,\mathrm{N\,m^{-2}}$ in the solar photosphere, and $Z_{\mathrm{II}} = 2.30$ for calcium. We are also given $\chi_{\mathrm{II}} = 11.9\,\mathrm{eV}$ and $Z_{\mathrm{III}} = 1$. With this, we can write the Saha equation for the ratio of doubly-ionized calcium to singly-ionized calcium:

$$\frac{N_{\mathrm{III}}}{N_{\mathrm{II}}} = \frac{2kTZ_{\mathrm{III}}}{P_e Z_{\mathrm{II}}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\mathrm{II}}/kT} = 0.002.$$

  Very little calcium is doubly ionized in the Sun's photosphere; instead, the calcium is almost all in the singly-ionized state—Example 8.1.5 shows that the ratio of singly-ionized calcium to neutral calcium is very high, and that almost all of the singly-ionized calcium is in the ground state. This means that almost all of the calcium in the Sun's photosphere can contribute to the formation of the H and K lines.

- Q8.14

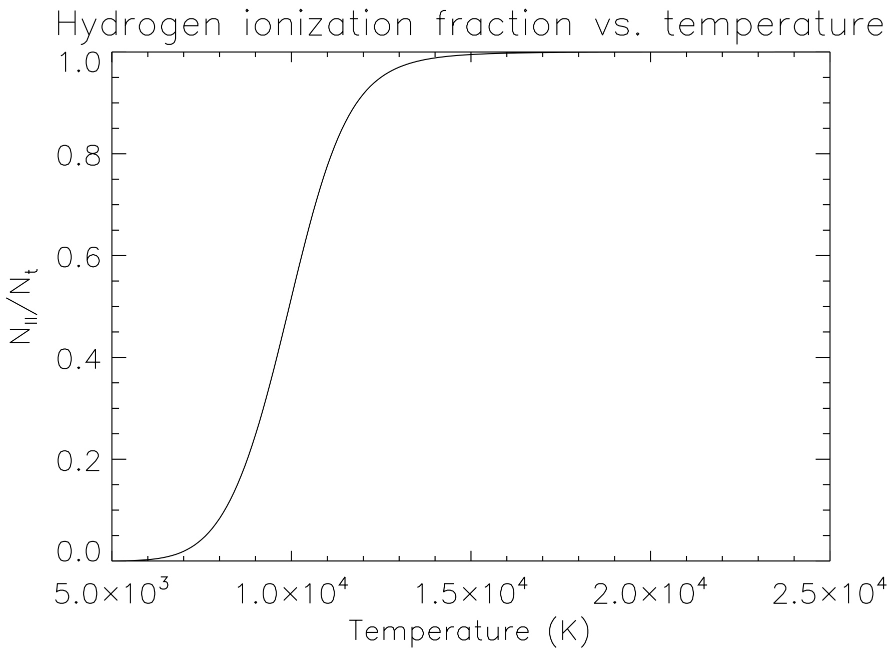
  The Saha equation is

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}.$$

  We can see that the ratio N_{i+1}/N_i increases as the temperature T increases, and that it decreases as the electron density n_e increases. If we consider a main-sequence star and a giant star of the same spectral type, the fact that they have the same spectral type means that they should have (nearly) the same ratio N_{i+1}/N_i for all species so that they form the spectral lines with the same relative strengths. The fact that the giant star has a lower atmospheric density means that it has a lower value of n_e than the main-sequence star, so in order to maintain the same value of N_{i+1}/N_i as the main-sequence star the giant star must also have a lower value of T than the main-sequence star.

- Q8.16

  Fomalhaut has $V = 1.19$; from the H-R diagram, it also has $M_V \approx 2$. We can solve for the distance to Fomalhaut using $V - M_V = 5\log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right)$, which gives $d \approx 7\,\mathrm{pc}$.


Hydrogen ionization fraction vs. temperature
1.0
0.8
0.6
0.4
0.2
0.0
N_{II}/N_t
5.0×10³
1.0×10⁴
1.5×10⁴
2.0×10⁴
2.5×10⁴
Temperature (K)