Homework Assignment 5 — Solutions

- Q7.1

  Recall from the definition of the center of mass that $m_1 m_1 \mathbf{r}_1 = -m_2 \mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are measured from the center of mass. This gives us $|\mathbf{r}_2| = \frac{m_1}{m_2}|\mathbf{r}_1|$; as a result, when star 1 is at its furthest distance from the center of mass, star 2 is also at its furthest distance from the center of mass. Remember that the furthest distance from focus 1 to a point on an ellipse occurs when an object is on its major axis near focus 2, at which point its distance from focus 1 is $a+ae$, where a and e are the semimajor axis and eccentricity of the ellipse; in addition, the center of mass is a shared focus of the two elliptical orbits, which tells us that star 1 is at a distance $a_1(1+e)$ from the center of mass when star 2 is at a distance $a_2(1+e)$ from the center of mass (note that we have assumed here that both stars orbit in ellipses with the same eccentricity; it's true, and if we wanted to prove it we could use our expressions for the stars' positions relative to the center of mass). At this moment, since the stars are always on opposite sides of the center of mass, we have $\mathbf{r}_1 = -a_1\hat{n}$ and $\mathbf{r}_2 = a_2\hat{n}$, where $\hat{n}$ is a unit vector that points from the center of mass towards the apoastron of star 1. Thus the corresponding position of the reduced mass in the associated one-body problem is $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (a_1 + a_2)(1+e)\hat{n}$. This is the maximum distance of the reduced mass from the origin in the associated one-body problem, because it is the maximum value of $\mathbf{r}$ (the maximum separation between stars 1 and 2). The distance is thus equal to $a(1+e)$, where a is the semimajor axis of the reduced mass's orbit. Setting $a(1+e) = (a_1 + a_2)(1+e)$ gives $a = a_1 + a_2$.

- Q7.3

  (a). From Fig. 7.8, the stars will *just* eclipse one another when the projection of their separation on the plane of the sky, $a\cos i$, equals the sum of their radii $r_1 + r_2$. Solving for the inclination,

  $$i = \cos^{-1}\left(\frac{r_1 + r_2}{a}\right).$$

  (b). For $a = 2\,\mathrm{AU} = 430\,\mathrm{R}_\odot$, $r_1 = 10\,\mathrm{R}_\odot$ and $r_2 = 1\,R_\odot$, the above expression gives $i = 1.55\,\mathrm{rad} = 88.5°$.

- Q7.4

  (a). The semimajor axis of the reduced-mass orbit *in AU* is given by

  $$a = \theta d$$

  where θ is the true angular extent of the semimajor axis *in arcseconds*, and d is the distance *in parsecs*. Given that $d = 1/p$, where p is the parallax in arcseconds, we have

  $$a = \theta/p.$$

  For Sirius, $\theta = 7.61''$ and $p = 0.379''$, meaning that $a = 20.1\,\mathrm{AU}$.
  From eqn. (2.37) of *Ostlie & Carroll*, we can write the general form of Kepler's third law as

  $$P^2 = a^3/M$$

  where P is the orbital period of the system in years, a is the semimajor axis of the reduced-mass orbit in AU, and M is the total mass of the system in solar units. Plugging in the value of a derived above, together with the measured period $P = 49.94\,\mathrm{yr}$, gives a total mass $M = 3.25\,\mathrm{M}_\odot$. To find the individual masses, we use the ratio of distances from the center of mass $a_A/a_B = m_B/m_A = 0.466$. This means that the total mass $M = m_A + m_B = 1.466\,m_A = 3.25\,\mathrm{M}_\odot$, so that $m_A = 2.22\,\mathrm{M}_\odot$ and $m_B = 1.03\,\mathrm{M}_\odot$.

(b). The absolute bolometric magnitudes of Sirius A and B are $M_A = +1.36$ and $M_B = +8.79$. The luminosities of Sirius A and B are related to their absolute bolometric magnitudes by

$$M - M_\odot = -2.5 \log_{10}\left(\frac{L}{L_\odot}\right).$$

Plugging in $M_\odot = +4.74$ gives $L_A = 22.5\,L_\odot$ and $L_B = 0.0240\,L_\odot$.

(c). $T_{\mathrm{eff},B} \approx 24,790\,\mathrm{K}$, and $L_B = 0.0240\,L_\odot = 9.34 \times 10^{24}\,\mathrm{W}$. We have $L_B = 4\pi r_B^2 \sigma T_{\mathrm{eff},B}^4$, which gives $r_B = 5.89 \times 10^6\,\mathrm{m} = 0.00847\,r_\odot = 0.925\,r_\oplus$. Sirius B has a mass slightly larger than the Sun, but a radius slightly smaller than the Earth.

- Q7.6

(a). We have

$$\frac{m_A}{m_B} = \frac{v_{Br}}{v_{Ar}} = \frac{22.4\,\mathrm{km/s}}{5.4\,\mathrm{km/s}} = 4.2.$$

(b). Assuming $i \approx 90^\circ$, the sum of the masses is given by

$$m_A + m_B = \frac{P}{2\pi G}\left(v_{Ar} + v_{Br}\right)^3 = \frac{6.31\,\mathrm{yr}}{4.2 \times 10^{-10}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}}\left(5.4\,\mathrm{km/s} + 22.4\,\mathrm{km/s}\right)^3 = 1.0 \times 10^{31}\,\mathrm{kg} = 5.0 M_\odot$$

(c). From (a) and (b), $m_A/m_B = 4.2$ and $m_A + m_B = 1.0 \times 10^{31}\,\mathrm{kg} = 5.0 M_\odot$. Since $m_A = 4.2\,m_B$, we have $5.2\,m_B = 5.0\,M_\odot$, or $m_B = 0.96\,M_\odot$. Then $m_A = 4.2\,m_B = 4.0\,M_\odot$.

(d). If we assume that the orbits are circular and continue to assume that $i \approx 90^\circ$, then $v_A = v_{Ar}$ and $v_B = v_{Br}$. We can assume that star A has larger radius than star B because it is more massive (this isn't always true: there are certain kinds of objects, such as white dwarfs and neutron stars, that have smaller radii for larger masses. If A and B were both white dwarfs, then the more massive star A would be smaller than star B. However, given the information in the problem, star A is too massive to be a white dwarf and probably too massive to be a neutron star). So $r_A > r_B$, which means that the time between first contact and minimum light is the length of time it takes to travel a distance $2\,r_B$ while moving at speed $v = v_A + v_B$ (the total relative speed of the two stars). That is,

$$r_B = \frac{v}{2}\left(t_b - t_a\right) = \frac{5.4\,\mathrm{km/s} + 22.4\,\mathrm{km/s}}{2}\left(0.58\,\mathrm{d}\right) = 7.0 \times 10^8\,\mathrm{m}.$$

Similarly, we have

$$r_A = r_B + \frac{v}{2}\left(t_c - t_b\right) = 7.0 \times 10^8\,\mathrm{m} + \frac{5.4\,\mathrm{km/s} + 22.4\,\mathrm{km/s}}{2}\left(0.64\,\mathrm{d}\right) = 1.5 \times 10^9\,\mathrm{m}.$$

(e). The apparent bolometric magnitudes at maximum, primary minimum and secondary minimum are $m_{\max} = +5.40$, $m_{\min,1} = +9.20$ and $m_{\min,2} = +5.44$. Since the distance to the system does not vary between maximum and the minima, we can use the apparent magnitudes to determine ratios of luminosities at different phases: $m_{\min,1} - m_{\max} = -2.5\log_{10}\left(\frac{L_{\min,1}}{L_{\max}}\right)$, $m_{\min,1} - m_{\min,2} = -2.5\log_{10}\left(\frac{L_{\min,1}}{L_{\min,2}}\right)$ and $m_{\min,2} - m_{\max} = -2.5\log_{10}\left(\frac{L_{\min,2}}{L_{\max}}\right)$. This gives $L_{\min,1}/L_{\max} = 0.030$, $L_{\min,1}/L_{\min,2} = 0.031$, and $L_{\min,2}/L_{\max} = 0.96$. Now we express the luminosities at various phases in terms of the radii and effective temperatures of the stars (ignoring effects due to interstellar dust/gas and the detector): at maximum we see all the light from both stars, so $L_{\max} = \pi r_A^2 \sigma T_{\mathrm{eff},A}^4 + \pi r_B^2 \sigma T_{\mathrm{eff},B}^4$. We can assume that star A, being more massive than star B, is also hotter (earlier spectral type)—the masses and sizes of the stars are consistent with main-sequence stars, so more massive goes with hotter surface temperature. Therefore, at secondary minimum we see all the light from star A only: $L_{\min,2} = \pi r_A^2 \sigma T_{\mathrm{eff},A}^4$. At primary minimum we see all the light from star B plus the light from star A reduced by the amount blocked by the disk of star B: $L_{\rm min,1} = \pi\left(r_A^2 - r_B^2\right)\sigma T_{\rm eff,A}^4 + \pi r_B^2 \sigma T_{\rm eff,B}^4$. The ratio of luminosities at maximum and secondary minimum is $L_{\rm max}/L_{\rm min,2} = 1 + \frac{r_B^2 T_{\rm eff,B}^4}{r_A^2 T_{\rm eff,A}^4}$; we also know that $L_{\rm max}/L_{\rm min,2} = 1/0.96$, so $\frac{r_B^2 T_{\rm eff,B}^4}{r_A^2 T_{\rm eff,A}^4} = 1/0.96 - 1 = 0.038$. Plugging in our values for r_A and r_B from part (d), we have $\frac{T_{\rm eff,B}^4}{T_{\rm eff,A}^4} = 0.17$, so $\frac{T_{\rm eff,B}}{T_{\rm eff,A}} = 0.64$.

Note that equation 7.11 in the book is derived for the case where the smaller star is hotter than the bigger star; in that case, secondary minimum occurs when the small star is in front of the big star and primary minimum occurs when the small star is completely hidden by the big star. Our case is the reverse: at secondary minimum the small star is completely hidden by the big star, and at primary minimum the small star is in front of the big star.

- Q7.7

From Fig. 7.2, the V-band magnitude at maximum is $V_{\rm max} \approx 10.04$, at primary minimum it is $V_{\rm min,1} \approx 10.78$, and at secondary minimum it is $V_{\rm min,2} \approx 10.68$. This gives luminosity ratios $L_{\rm min,1}/L_{\rm max} = 0.51$, $L_{\rm min,1}/L_{\rm min,2} = 0.91$, and $L_{\rm min,2}/L_{\rm max} = 0.55$. If we assume the larger star has radius r_l and effective temperature $T_{\rm eff,l}$, while the smaller star has radius r_s and effective temperature $T_{\rm eff,s}$, then the luminosity at maximum is $L_{\rm max} = \pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4$, the luminosity at secondary minimum is $L_{\rm min,2} = \pi r_l^2 \sigma T_{\rm eff,l}^4$, and the luminosity at primary minimum is $L_{\rm min,1} = \pi\left(r_l^2 - r_s^2\right)\sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4$. This assumes that the bigger star is also hotter; it looks like both stars in YY Sag are main-sequence stars, so this assumption should be correct. This gives us

$$\frac{L_{\rm min,2}}{L_{\rm max}} = \frac{\pi r_l^2 \sigma T_{\rm eff,l}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 1 - \frac{\pi r_s^2 \sigma T_{\rm eff,s}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 0.55$$

and

$$\frac{L_{\rm min,1}}{L_{\rm max}} = \frac{\pi\left(r_l^2 - r_s^2\right)\sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 1 - \frac{\pi r_s^2 \sigma T_{\rm eff,l}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 0.51.$$

This gives

$$\frac{\pi r_s^2 \sigma T_{\rm eff,s}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 0.45$$

and

$$\frac{\pi r_s^2 \sigma T_{\rm eff,l}^4}{\pi r_l^2 \sigma T_{\rm eff,l}^4 + \pi r_s^2 \sigma T_{\rm eff,s}^4} = 0.49.$$

Taking the ratio of these two expressions gives

$$\frac{\pi r_s^2 \sigma T_{\rm eff,s}^4}{\pi r_s^2 \sigma T_{\rm eff,l}^4} = \frac{T_{\rm eff,s}^4}{T_{\rm eff,l}^4} = \frac{0.45}{0.49},$$

so that $T_{\rm eff,s}/T_{\rm eff,l} = 0.98$.

If instead we assumed that the smaller star is hotter, we would get $T_{\rm eff,s}/T_{\rm eff,l} = 1.02$. This would imply that the stars are not both main-sequence stars, since main-sequence stars increase in temperature and radius with mass, which means that larger main-sequence stars are also hotter.

- Q7.13

  The observed luminosity from the Sun when it is not eclipsed is $\pi r_\odot^2 \sigma T_{\text{eff},\odot}^4$. When Jupiter passes in front of the Sun, it blocks an area of size πr_J^2, and the observed luminosity decreases to $\pi \left(r_\odot^2 - r_\text{J}^2\right) \sigma T_{\text{eff},\odot}^4$. The fractional decrease in the observed brightness is

$$\frac{\pi \left(r_\odot^2 - r_\text{J}^2\right) \sigma T_{\text{eff},\odot}^4}{\pi r_\odot^2 \sigma T_{\text{eff},\odot}^4} - 1 = -\frac{r_\text{J}^2}{r_\odot^2} \approx -0.01.$$

  The eclipse only reduces the brightness by about 1%.

- Q7.15

  (a). See plots at the end of the solutions.

  (b). From section 7.3, we know that $\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}}$ and $m_1 + m_2 = \frac{P}{2\pi G} \frac{(v_{1r}+v_{2r})^3}{\sin^3 i}$. Since we know m_1, m_2, P, and i we can solve for the amplitudes of v_{1r} and v_{2r} (we already know that v_{1r} and v_{2r} should vary sinusoidally with time, which is borne out by the radial velocity curves in the $e = 0$ plot). The result is that $v_{1r} = 13000\,\text{m/s}$ and $v_{2r} = 3300\,\text{m/s}$, which agrees with the amplitudes of the sinusoidal velocity curves in the $e = 0$ plot.

  (c). As the eccentricity increases, the shape of the velocity curves goes from sinusoidal to a flattened "top-hat" shaped curve; we can estimate the system eccentricity by examining the shape of the velocity curves.

- Q7.18

  See plot at the end of the solutions.

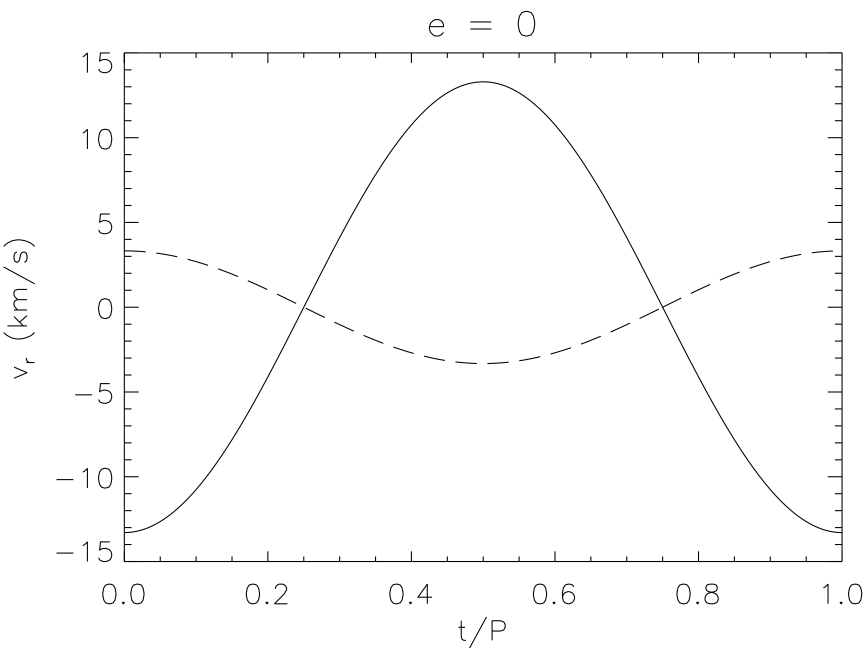

Figure 1: Radial velocity curves (solid – primary; dashed – secondary) for the $e = 0$ case in Q7.15(a).

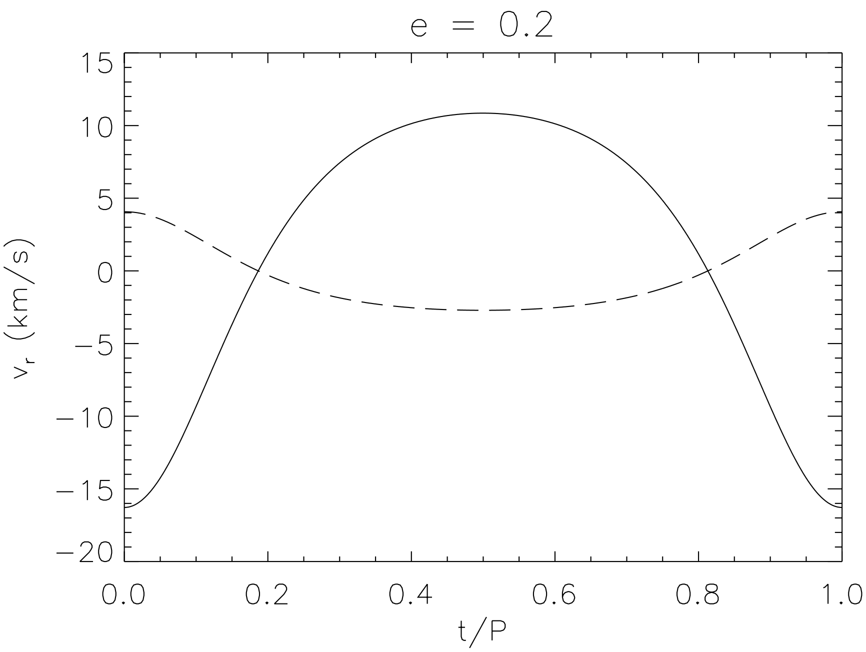

Figure 2: Radial velocity curves (solid – primary; dashed – secondary) for the $e = 0.2$ case in Q7.15(a).

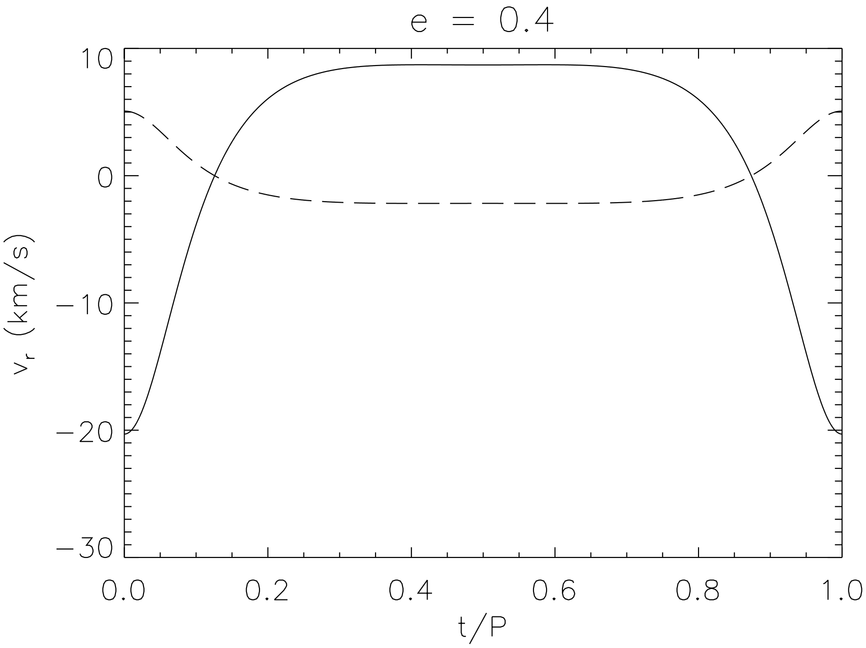

Figure 3: Radial velocity curves (solid – primary; dashed – secondary) for the $e = 0.4$ case in Q7.15(a).

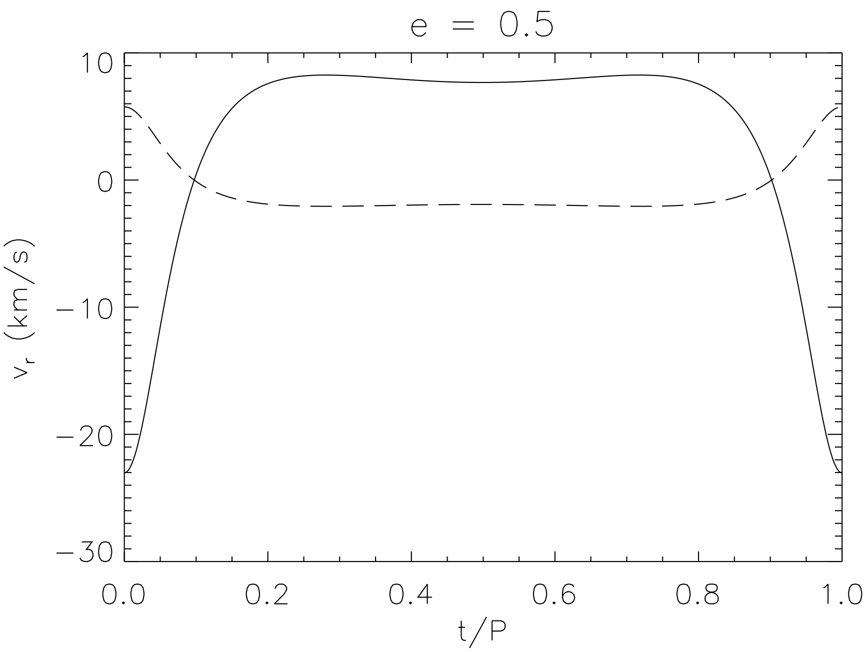

Figure 4: Radial velocity curves (solid – primary; dashed – secondary) for the $e = 0.5$ case in Q7.15(a).

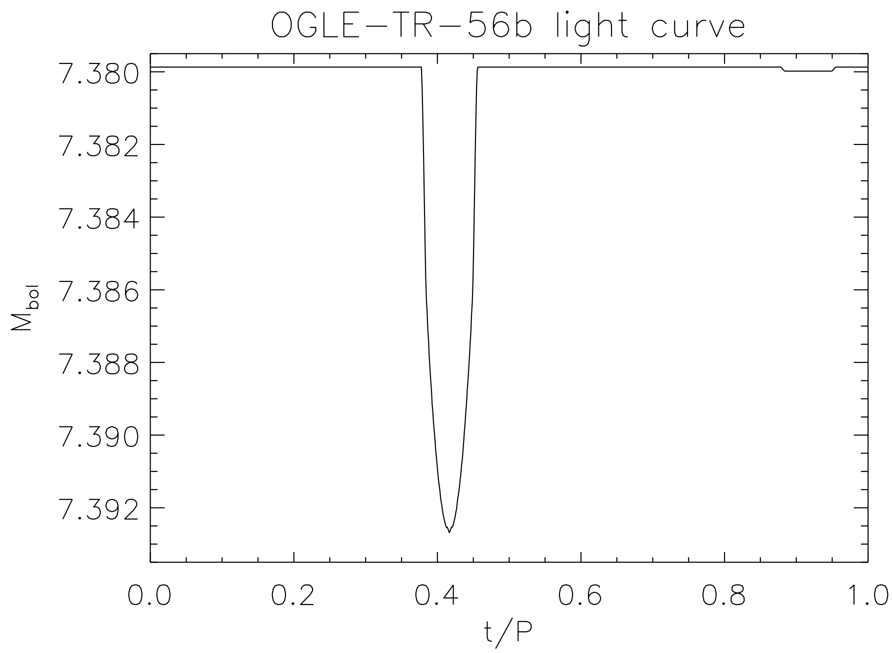

Figure 5: Light curve for OGLE-TR-56b in Q7.18.