Homework Assignment 4 — Solutions

- Extra 1

  In general, we have

$$\nu_{\text{obs}} = \nu_{\text{rest}} \frac{\sqrt{1-u^2/c^2}}{1+(u/c)\cos\theta} = \nu_{\text{rest}} \frac{\sqrt{1-u^2/c^2}}{1+(v_r/c)} \tag{1}$$

For pure radial motion, $\theta = 0°$ or $180°$, and $u = v_r$ or $u = -v_r$. In this case, we have

$$\nu_{\text{obs}} = \nu_{\text{rest}} \frac{\sqrt{1-v_r^2/c^2}}{1+v_r/c}.$$

Factoring $1 - v_r^2/c^2 = (1+v_r/c)(1-v_r/c)$ and plugging in, we have

$$\nu_{\text{obs}} = \nu_{\text{rest}} \sqrt{\frac{1-v_r/c}{1+v_r/c}}.$$

Using the Taylor series approximation $(1+x)^{-1} \approx 1 - x$ on the denominator (inside the square root), we have

$$\nu_{\text{obs}} \approx \nu_{\text{rest}} \sqrt{(1-v_r/c)(1-v_r/c)} = \nu_{\text{rest}}(1-v_r/c).$$

Switching from frequency to wavelength, we have

$$\frac{c}{\lambda_{\text{obs}}} \approx \frac{c}{\lambda_{\text{rest}}}(1-v_r/c),$$

or

$$\lambda_{\text{obs}} \approx \lambda_{\text{rest}}(1-v_r/c)^{-1}.$$

Now, using the Taylor series approximation $(1-x)^{-1} \approx 1 + x$, we have

$$\lambda_{\text{obs}} \approx \lambda_{\text{rest}}(1+v_r/c).$$

Subtracting λ_{rest} from both sides, we have

$$\lambda_{\text{obs}} - \lambda_{\text{rest}} = \Delta\lambda \approx \lambda_{\text{rest}} \frac{v_r}{c}.$$

Dividing through by λ_{rest}, we get our result:

$$\frac{\Delta\lambda}{\lambda_{\text{rest}}} = z \approx \frac{v_r}{c}.$$

The Taylor series approximations we used are all for $x \ll 1$, so in our case they only work if $v_r \ll c$.

For non-radial motion, we go back to equation 1. For a star moving with $\frac{v_r}{c} = 10^{-4}$ we can use our approximation $(1+x)^{-1} \approx 1 - x$ on the denominator to get

$$\nu_{\text{obs}} = \nu_{\text{rest}} \frac{\sqrt{1-u^2/c^2}}{1+(v_r/c)} \approx \nu_{\text{rest}} \left(\sqrt{1-u^2/c^2}\right)(1-(v_r/c))$$

If $u \ll c$ (we'll check this assumption later*) then we can approximate the square root using $\sqrt{(1-x)} \approx 1 - \frac{x}{2}$:

$$\nu_{\rm obs} \approx \nu_{\rm rest}\left(1 - \frac{u^2}{2c^2}\right)(1 - (v_r/c)).$$

Switching from frequency to wavelength, we have

$$\lambda_{\rm obs} \approx \lambda_{\rm rest}\left(1 - \frac{u^2}{2c^2}\right)^{-1}(1 - (v_r/c))^{-1},$$

and using $(1-x)^{-1} \approx 1 + x$ twice we get

$$\lambda_{\rm obs} \approx \lambda_{\rm rest}\left(1 + \frac{u^2}{2c^2}\right)(1 + (v_r/c)) = \lambda_{\rm rest}\left(1 + \frac{v_r}{c} + \frac{u^2}{2c^2} + \frac{v_r u^2}{2c^3}\right).$$

This gives

$$\frac{\Delta\lambda}{\lambda_{\rm rest}} = z \approx \left(\frac{v_r}{c} + \frac{u^2}{2c^2} + \frac{v_r u^2}{2c^3}\right). \qquad (2)$$

We can see that our previous approximation $z \approx v_r/c$ is good if $v_r/c \gg \frac{u^2}{2c^2}$—in that case, we can drop the second and third terms inside the parentheses on the right hand side of equation 2. Conversely, our first-order approximation breaks down if $\frac{u^2}{2c^2}$ becomes comparable to v_r/c. Using $v_r = u\cos\theta$, we can see that the approximation breaks down if $|\cos\theta| \lesssim u/2c$; since $u = v_r/\cos\theta$ this happens when $\cos\theta \lesssim \frac{v_r}{2\,c\cos\theta}$, or when $\cos^2\theta \lesssim v_r/2c$. For a star with $\frac{v_r}{c} = 10^{-4}$, the first-order approximation breaks down when $\cos^2\theta \lesssim 5 \times 10^{-5}$, which happens for angles θ between 89.6° and 90.4°—in other words, the direction of motion has to be within about 0.4° of exactly perpendicular to the line of sight for the first-order approximation to break down. Since stellar motions are generally oriented randomly with respect to the line of sight, we can see that the first-order approximation is almost always good, as long as $v_r \ll c$.

* Checking the assumption that $u \ll c$: we have $u = v_r/\cos\theta$ and $v_r/c = 10^{-4}$, so $u \ll c$ as long as $\cos\theta \gg 10^{-4}$. This is automatically true if $\cos^2\theta > 5 \times 10^{-5}$, so if our first-order approximation is good at all then the assumption $u \ll c$ is also good and the analysis presented above works fine. More generally, for other values of v_r/c, as long as $v_r/c \ll 1$ the assumption that $u/c \ll 1$ doesn't affect the analysis of what (small) range of angles θ gives a break down of the first-order approximation.

- Q4.8

  $\lambda_{\rm rest} = 121.6$ nm, $\lambda_{\rm obs} = 656.8$ nm. We have $\Delta\lambda = \lambda_{\rm obs} - \lambda_{\rm rest} = 535.2$ nm, and $z = \Delta\lambda/\lambda_{\rm rest} = 4.401$. Since $z \not\ll 1$, we can't use the first-order approximation $z \approx \frac{v_r}{c}$ (if we did, we'd get a faster-than-light velocity for the quasar, which, despite recent unexplained neutrino experiment results, is a no-no). Instead, we need to use the equation

$$\nu_{\rm obs} = \nu_{\rm rest}\sqrt{\frac{1 - v_r/c}{1 + v_r/c}}.$$

  We are justified in using this radial velocity expression for the relativistic Doppler shift because we expect that the quasar's motion is almost exactly radial, with negligible transverse motion—the radial motion comes from the cosmic expansion, and should be much larger than the peculiar (or transverse) motion of the quasar. Switching from frequency to wavelength, we have

$$\lambda_{\rm obs} = \lambda_{\rm rest}\sqrt{\frac{1 + v_r/c}{1 - v_r/c}},$$

or

$$z = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1.$$

Solving for v_r/c gives

$$\frac{v_r}{c} = \frac{(z+1)^2 - 1}{(z+1)^2 + 1}.$$

Plugging in $z = 4.401$ gives $v_r = 0.93\,c$.

- Q5.1

  Barnard's star has $\mu = 10.3577''/\text{yr}$, $p = 0.54901''$, and $\lambda_{\text{H}\alpha,\text{obs}} = 656.034$ nm.

  (a). $\lambda_{\text{H}\alpha,\text{rest}} = 656.469$ nm, so $z = \lambda_{\text{H}\alpha,\text{obs}}/\lambda_{\text{H}\alpha,\text{rest}} - 1 = -6.626 \times 10^{-4}$. Since $|z| \ll 1$, we can use our simple first-order approximation $z = v_r/c$, giving $v_r = -1.988 \times 10^5$ m/s.

  (b). The distance to Barnard's star is $d = 1/0.54901\,\text{pc} = 1.821\,\text{pc}$, and its transverse velocity is $v_t = \mu\,d = (10.3577''/\text{yr})(1.821\,\text{pc}) = (1.591 \times 10^{-12}\,\text{rad/s})(5.619 \times 10^{16}\,\text{m}) = 8.94 \times 10^4\,\text{m/s}$.

  (c). The total speed of Barnard's star through space is $v_{\text{Barnard'sstar}} = (v_r^2 + v_t^2)^{1/2} = 2.18 \times 10^5\,\text{m/s}$.

- Q5.2

  We have $\lambda_1 = 588.997$ nm and $\lambda_2 = 589.594$ nm.

  (a). The grating has 300 lines/mm, so the distance between adjacent lines is $d = 1/300$ mm $=$ 0.0003333 cm. The angles of the two lines in the second-order spectrum are given by $\sin\theta_1 = 2\lambda_1/d$ and $\sin\theta_2 = 2\lambda_2/d$, or $\theta_1 = 0.003538$ rad $= 0.2027°$ and $\theta_2 = 0.003534$ rad $= 0.2025°$. The difference in the angles is $\Delta\theta = \theta_1 - \theta_2 = 4 \times 10^{-6}$ rad $= 0.0002°$.

  (b). In order to resolve the lines, we must have $\frac{\lambda}{nN} \leq \Delta\lambda$. We have $\Delta\lambda = 0.597$ nm and $\lambda = 588.997$ nm (or $\lambda = 589.594$ nm, it doesn't really matter). If we consider the second-order spectrum ($n = 2$), then in order to resolve the lines we must have $\frac{1}{N} \leq \frac{n\Delta\lambda}{\lambda} = 0.00203$. Flipping this gives $N \geq 493.3$; since we cannot illuminate a fractional number of lines, we must have N = 494 (or more).

- Q5.11

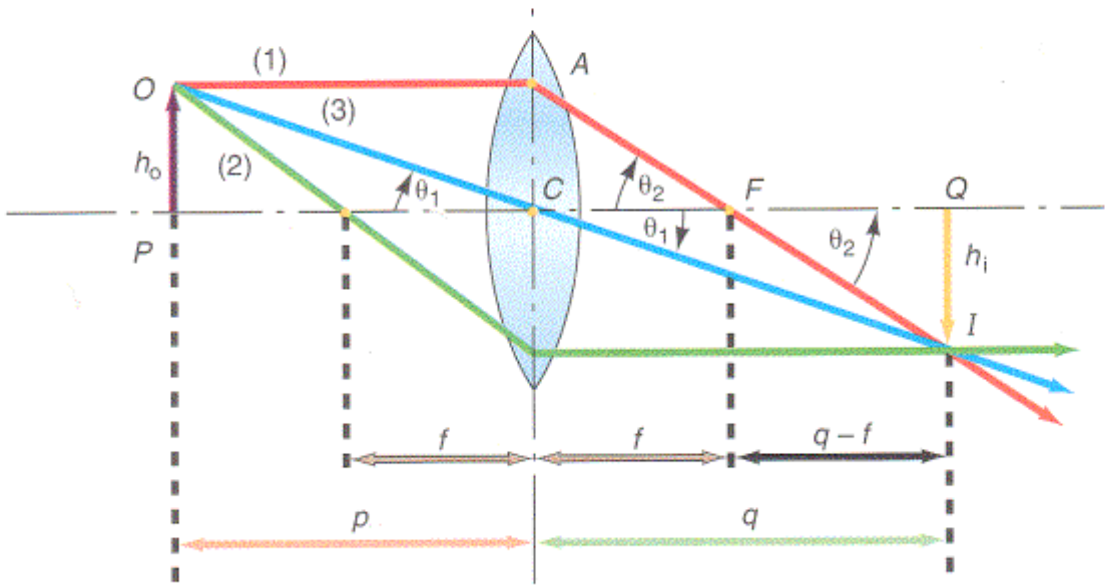
  The limit wavelengths for the Lyman, Balmer and Paschen series can be calculated from

  $$\frac{1}{\lambda_{\text{limit}}} = R_H \frac{1}{m^2},$$

  where $m = 1, 2, 3$. This gives $\lambda_{\text{Ly,limit}} = 91.2$ nm, $\lambda_{\text{Ba,limit}} = 365$ nm, and $\lambda_{\text{Pa,limit}} = 821$ nm. These values agree closely with the values reported in Table 5.2 of the text—note that the values reported for the Balmer and Paschen limits are the wavelength measured in air, which is very slightly different from the vacuum wavelength. The Lyman and Balmer limits are in the ultraviolet, and the Paschen limit is in the infrared.

- Q6.2

  (a). The figure shows a third (green) ray that goes from the tip of the arrow through the focal point on the near side of the lens; this ray bends through the lens so that it leaves the lens traveling parallel to the optical axis, and it pass through the tip of the image arrow, just as the blue and red rays do. We can derive the lens equation just using the first two rays, as follows: we can see that triangle OPC is similar to triangle IQC, which gives us the relation $h_\text{o}/p = h_\text{i}/q$. We can also see that triangle ACF is similar to triangle IQF, so that $h_\text{o}/f = h_\text{i}/(q - f)$. We can combine the two relations to give $h_\text{i}p/fq = h_\text{i}/(q - f)$. Dividing through by h_i gives $p/fq = 1/(q - f)$; multiplying through by $(q - f)/p$ gives $(q - f)/fq = 1/p$, or $1/f - 1/q = 1/p$, which finally gives the desired result: $1/f = 1/p + 1/q$.



(b). When $p \gg f$, the lens equation can be approximated as $1/f \approx 1/q$, which shows that the image is located at a distance $q \approx f$, meaning that the image lies on the focal plane.

- Q6.8

(a). $D = 20$ cm, $\lambda = 550$ nm. The Rayleigh criterion gives a diffraction-limited resolution of $\Delta\theta = 1.22\,\lambda/D = 3.36 \times 10^{-6}$ rad $= 1.92 \times 10^{-4\,\circ} = 0.692''$.

(b). From Appendix C, the distance to the Moon is $d = 384,000$ km $= 3.84 \times 10^8$ m. A crater that subtends an angle $\Delta\theta = 3.36 \times 10^{-6}$ rad at that distance will have diameter $D_{\text{crater}} = d\,\Delta\theta = 1290$ m $= 1.29$ km.

(c). The telescope's theoretical diffraction limit for angular resolution will probably not be achieved: seeing typically limits the angular resolution to $1''$ or worse in most locations, making the theoretical diffraction limit of $0.692''$ irrelevant.

- Q6.15

SIM PlanetQuest will have angular resolution $\Delta\theta \lesssim 0.000004''$ down to $V = 20$.

(a). At a distance $d = 10$ km, SIM can detect differences Δl in the length of a grass blade as small as $\Delta l = d\,\Delta\theta = 2 \times 10^{-7}$ m, or 200 nm. If grass grows 2 cm/week, SIM can measure growth over a period as short as $\Delta t = 2 \times 10^{-7}\,\text{m}/(0.02\,\text{m/week}) = 10^{-5}$ weeks $= 6$ seconds.

(b). The maximum distance in parsecs that SIM can measure using trigonometric parallax is one over its angular resolution in arcseconds: $d_{\max} = (1/0.000004)\,\text{pc} = 250\,\text{kpc}$, or about 30 times the distance from the Sun to the center of the Milky Way.

(c). The Sun has absolute magnitude $M_\odot = +4.77$. If we imagine looking at the Sun from a distance of 250 kpc, its apparent magnitude would be $m_{\odot,250\,\text{kpc}} = M_\odot + 5\log_{10}\left(\frac{250\,\text{kpc}}{10\,\text{pc}}\right) = +26.8$.

(d). Sirius has absolute magnitude $M_{\text{Sirius}} = -5.14$. Sirius would have an apparent magnitude $m_{\text{Sirius}} = 20$ at a distance d such that $m_{\text{Sirius}} - M_{\text{Sirius}} = 5\log_{10}\left(\frac{d}{10\,\text{pc}}\right)$. Solving gives $d = 1.1 \times 10^6$ pc.