# Homework Assignment 2 — Solutions

- Q2.1

  Transforming from a polar coordinate system $(r, \theta)$ with its origin at one focus to a rectangular coordinate system with its origin at the center of the ellipse, we have:

$$x = a\,e + r\,\cos\theta,\;\; y = r\,\sin\theta$$

where $a$ is the ellipse's semimajor axis and $e$ is its eccentricity. Note that $r$ is a function of $\theta$ (so, for instance, the maximum value of $y$ doesn't occur right at $\theta = \pi/2$ radians)–specifically,

$$r(\theta) = \frac{a(1-e^2)}{1+e\,\cos\theta}.$$

Plugging in our expression for $r(\theta)$ gives

$$x = a\,e + \frac{a(1-e^2)}{1+e\,\cos\theta}\,\cos\theta$$

and

$$y = \frac{a(1-e^2)}{1+e\,\cos\theta}\,\sin\theta.$$

Using $b = a\sqrt{(1-e^2)}$, we have

$$\frac{x}{a} = e + \frac{(1-e^2)}{1+e\,\cos\theta}\,\cos\theta$$

and

$$\frac{y}{b} = \frac{\sqrt{(1-e^2)}}{1+e\,\cos\theta}\,\sin\theta,$$

so

$$\begin{aligned}
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= e^2 + 2\,e\frac{(1-e^2)}{1+e\,\cos\theta}\,\cos\theta + \frac{(1-e^2)^2}{(1+e\,\cos\theta)^2}\,\cos^2\theta + \frac{(1-e^2)}{(1+e\,\cos\theta)^2}\,\sin^2\theta \\
&= \frac{1}{(1+e\,\cos\theta)^2}\left(e^2\,(1+e\,\cos\theta)^2 + 2\,e\,(1-e^2)\,(1+e\,\cos\theta)\,\cos\theta + (1-e^2)^2\,\cos^2\theta + (1-e^2)\,\sin^2\theta\right. \\
&= \frac{1}{(1+e\,\cos\theta)^2}\left(e^2 + 2\,e^3\,\cos\theta + e^4\,\cos^2\theta + 2\,e\,\cos\theta + 2\,e^2\,\cos^2\theta - 2\,e^3\,\cos\theta - 2\,e^4\,\cos^2\theta + \right. \\
&\quad \left.\cos^2\theta - 2\,e^2\,\cos^2\theta + e^4\,\cos^2\theta + \sin^2\theta - e^2\,\sin^2\theta\right) \\
&= \frac{1}{(1+e\,\cos\theta)^2}\left(e^4\left[\cos^2\theta - 2\,\cos^2\theta + \cos^2\theta\right] + e^3\,[2\,\cos\theta - 2\,\cos\theta] + \right. \\
&\quad \left. e^2\left[1 + 2\,\cos^2\theta - 2\,\cos^2\theta - \sin^2\theta\right] + 2\,e\,\cos\theta + \cos^2\theta + \sin^2\theta\right) \\
&= \frac{1}{(1+e\,\cos\theta)^2}\left(e^2\left[1-\sin^2\theta\right] + 2\,e\,\cos\theta + 1\right) \\
&= \frac{1}{(1+e\,\cos\theta)^2}\left(e^2\,\cos^2\theta + 2\,e\,\cos\theta + 1\right) \\
&= \frac{(1+e\,\cos\theta)^2}{(1+e\,\cos\theta)^2} \\
&= 1,
\end{aligned}$$

which is the desired result.

- Q2.2

  In Cartesian coordinates, the equation of an ellipse with its semimajor axis aligned with the x-axis is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

  The area of the ellipse is

$$A = \int_{x=-a}^{x=+a} \int_{y_{\min}(x)}^{y_{\max}(x)} dx\, dy,$$

  where $y_{\min}(x)$ and $y_{\max}(x)$ are the minimum and maximum values of $y$ at a given $x$. The points $(x, y_{\min}(x))$ and $(x, y_{\max}(x))$ lie on the circumference of the ellipse. Intuitively, the double integral represents adding up the areas of thin vertical strips stretching from the bottom of the ellipse to its top, adding up strips from the left end of the ellipse ($x = -a$) to the right end ($x = +a$). Using the equation of the ellipse we have

$$y_{\min}(x) = -b\left(1 - \frac{x^2}{a^2}\right)^{1/2}, \quad y_{\max}(x) = +b\left(1 - \frac{x^2}{a^2}\right)^{1/2},$$

  so

$$\begin{aligned} A &= \int_{x=-a}^{x=+a} \int_{-b\left(1-\frac{x^2}{a^2}\right)^{1/2}}^{+b\left(1-\frac{x^2}{a^2}\right)^{1/2}} dx\, dy \\ &= \int_{x=-a}^{x=+a} 2\, b\left(1 - \frac{x^2}{a^2}\right)^{1/2} dx. \end{aligned}$$

  Let $\sin u = \frac{x}{a}$. Then $\cos\, u\, du = \frac{1}{a}\, dx$ and the integral becomes

$$\begin{aligned} A &= 2\, b \int_{u=-\pi/2}^{u=+\pi/2} \left(1 - \sin^2\, u\right)^{1/2} a\, \cos\, u\, du \\ &= 2\, a\, b \int_{-\pi/2}^{\pi/2} \cos^2\, u\, du \\ &= 2\, a\, b \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos\,(2\, u)\right] du \\ &= 2\, a\, b \left[\frac{u}{2} + \frac{1}{4}\sin\,(2\, u)\right]_{-\pi/2}^{\pi/2} \\ &= 2\, a\, b \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) + \frac{1}{4}\sin\pi - \frac{1}{4}\sin\,(-\pi)\right] \\ &= 2\, a\, b \left[\frac{\pi}{2}\right] \\ &= \pi\, a\, b, \end{aligned}$$

  which is the desired result.

- Extra 1

  From problem 2.2, we know that the area of an ellipse is $A = \pi\, a\, b$. Using our old polar coordinate system $(r, \theta)$ with its origin at one focus, we can express the area of the ellipse as

$$A = \int_{\theta=0}^{\theta=2\,\pi} \frac{1}{2} r^2\,(\theta)\; d\theta.$$

We also know that

$$r(\theta) = \frac{a(1-e^2)}{1+e\,\cos\theta}$$

and

$$b = a\sqrt{(1-e^2)},$$

so we have

$$A = \frac{1}{2} a^2\,\left(1-e^2\right)^2 \int_{\theta=0}^{\theta=2\,\pi} \frac{d\theta}{\left(1+e\,\cos\,\theta\right)^2} = \pi\,a\,b = \pi\,a^2\,\left(1-e^2\right)^{1/2}.$$

Dividing through, we have

$$\int_0^{2\,\pi} \frac{d\theta}{\left(1+e\,\cos\,\theta\right)^2} = 2\,\pi\,\left(1-e^2\right)^{-3/2}.$$

- Q2.8

  (a). The Moon has semimajor axis $a_{\mathrm{Moon}} = 3.8 \times 10^8$ m and an orbital period $P_{\mathrm{Moon}} = 27$ days. The Hubble Space Telescope has $a_{\mathrm{HST}} = 610$ km $= 6.1 \times 10^5$ m. Using Kepler's third law,

  $$\left(\frac{P_{\mathrm{HST}}}{P_{\mathrm{Moon}}}\right)^2 = \left(\frac{a_{\mathrm{HST}}}{a_{\mathrm{Moon}}}\right)^3.$$

  Solving gives $P_{\mathrm{HST}} = 2.5$ minutes.

  (b). A geosynchronous orbit has period $P_{\mathrm{geo}} = 1$ day; using Kepler's third law again, we have

  $$\left(\frac{P_{\mathrm{geo}}}{P_{\mathrm{Moon}}}\right)^2 = \left(\frac{a_{\mathrm{geo}}}{a_{\mathrm{Moon}}}\right)^3,$$

  which gives $a_{\mathrm{geo}} = 4.2 \times 10^7$ m.

  (c). A geosynchronous orbit that maintains a satellite always above a fixed point on the Earth's surface (sometimes called a geostationary orbit) is only possible for an orbit above the Earth's equator*. An orbit that passes above a point north of the equator has an orbital plane that is tilted with respect to the equator, so the orbit passes above points north and south of the equator and a satellite in such an orbit can't remain always above a fixed point on the Earth's surface.

  * Actually, even an orbit above the Earth's equator can't be a perfect geostationary orbit because the satellite's motion is perturbed by the gravitational pull of the Moon, Sun, other planets, etc. A satellite that wants to remain always above a fixed point on Earth must periodically burn some fuel to correct its orbit.

- Q2.12

  (a). See Fig. 1

  (b). The best-fit line has equation $\log_{10}\left(\frac{P}{\mathrm{days}}\right) = -3.7 + 1.5\log_{10}\left(\frac{a}{10^3\mathrm{km}}\right)$, so it has slope 1.5.

(c). The best-fit line is $\log_{10}\left(\frac{P}{\text{days}}\right) = -3.7 + 1.5\log_{10}\left(\frac{a}{10^3\text{km}}\right)$, which implies $\left(\frac{P}{\text{days}}\right) = 10^{-3.7}\left(\frac{a}{10^3\text{km}}\right)^{1.5}$, or $\left(\frac{P}{\text{days}}\right)^2 = 10^{-7.4}\left(\frac{a}{10^3\text{km}}\right)^3$. Kepler's third law states that

$$P^2 = \frac{4\,\pi^2}{G\,(m_1 + m_2)}\,a^3.$$

Since the masses of the Galilean moons are all much less than the mass of Jupiter, we can approximate $m_1 + m_2 \approx M_{\text{Jup}}$. We have

$$P^2 = 10^{-7.4}\,\frac{\text{days}^2}{\left(10^3\text{km}\right)^3}\,a^3 \approx \frac{4\,\pi^2}{GM_{\text{Jup}}}\,a^3,$$

or

$$10^{-7.4}\,\frac{\text{days}^2}{\left(10^3\text{km}\right)^3} \approx \frac{4\,\pi^2}{GM_{\text{Jup}}}.$$

Solving gives $M_{\text{Jup}} \approx 2 \times 10^{27}$ kg.

- Q2.14

$P_{\text{Halley}} = 76$ yr and $e_{\text{Halley}} = 0.9673$.

(a). Using Kepler's third law to compare Halley's comet and the Earth gives

$$\left(\frac{P_{\text{Halley}}}{1\,\text{year}}\right)^2 = 76^2 = \left(\frac{a_{\text{Halley}}}{1\,\text{AU}}\right)^3,$$

which gives $a_{\text{Halley}} = 18\,\text{AU} = 2.7 \times 10^{12}$ m.

(b). Kepler's third law for Halley's comet is

$$P_{\text{Halley}}^2 = \frac{4\,\pi^2}{G\,(M_\odot + M_{\text{Halley}})}\,a_{\text{Halley}}^3 \approx \frac{4\,\pi^2}{G\,M_\odot}\,a_{\text{Halley}}^3.$$

Solving for $M_\odot$ gives

$$M_\odot = \frac{4\,\pi^2}{G\,P_{\text{Halley}}^2}\,a_{\text{Halley}}^3 = 2 \times 10^{30}\,\text{kg}.$$

(c). The perihelion distance is $a - ae = 0.59\,\text{AU} = 8.8 \times 10^{10}$ m.
The aphelion distance is $a + ae = 35\,\text{AU} = 5.3 \times 10^{12}$ m.

(d). In general, the orbital velocity satisfies

$$v^2 = G\,M_\odot\left(\frac{2}{r} - \frac{1}{a}\right).$$

Plugging in our values for $a$ from part a and for $r_{\text{perihelion}}$ and $r_{\text{aphelion}}$ from part c gives $v_{\text{perihelion}} = 1.7 \times 10^6$ m/s and $v_{\text{aphelion}} = 2.9 \times 10^4$ m/s. The radial distance on the semiminor axis of the orbit is $r_{\text{semimajor}} = \left(b^2 + [a\,e]^2\right)^{1/2} = \left(a^2\,\left[1 - e^2\right] + a^2\,e^2\right)^{1/2} = \left(a^2\right)^{1/2} = a$, so $v_{\text{semiminor}}^2 = \frac{G\,M_\odot}{a}$, which gives $v_{\text{semiminor}} = 2.2 \times 10^5$ m/s. It turns out the orbital velocity on the semiminor axis is the geometric mean of the orbital velocities at perihelion and aphelion.

- Q2.15

  Using Prof. Townsend's online interface for Orbit, I find that Halley's comet is 1 AU away from the principal focus at a time 0.107 years after perihelion. Note that Orbit has bad numerical resolution properties; in order to get the answer you need to use a large number of timesteps. It's a good idea to check whether you are using enough timesteps by doing a test run with twice as many timesteps; if you get the same answer after doubling the number of timesteps you should be in good shape.

- Extra 2

  We begin with the radial equation of motion

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{G\,M\,\mu}{r^2}, \tag{1}$$

and conservation of angular momentum

$$L = \mu\, r^2\, \dot{\theta}, \tag{2}$$

which can be rearranged to read

$$\dot{\theta} = \frac{L}{\mu\, r^2}. \tag{3}$$

Plugging this into equation 1 gives

$$\mu\left(\ddot{r} - r\left[\frac{L}{\mu\, r^2}\right]^2\right) = \mu\left(\ddot{r} - \frac{L^2}{\mu^2\, r^3}\right) = -\frac{G\,M\,\mu}{r^2}. \tag{4}$$

Next, we get rid of time derivatives in the radial equation of motion using the identity

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{L}{\mu\, r^2}\frac{d}{d\theta}, \tag{5}$$

where we used equation 3 in the last step.

Substituting into equation 4 gives

$$\mu\left(\frac{L}{\mu\, r^2}\frac{d}{d\theta}\left[\frac{L}{\mu\, r^2}\frac{dr}{d\theta}\right] - \frac{L^2}{\mu^2\, r^3}\right) = -\frac{G\,M\,\mu}{r^2}. \tag{6}$$

Now we make the substitution $u = \frac{1}{r}$, so that $dr = -\frac{1}{u^2}\,du$. Plugging into equation 6 gives

$$\mu\left(\frac{L\,u^2}{\mu}\frac{d}{d\theta}\left[-\frac{L\,u^2}{\mu}\frac{1}{u^2}\frac{du}{d\theta}\right] - \frac{L^2\,u^3}{\mu^2}\right) = -\frac{L^2\,u^2}{\mu}\frac{d^2u}{d\theta^2} - \frac{L^2\,u^3}{\mu} = -G\,M\,\mu\,u^2. \tag{7}$$

Multiplying through by $-\frac{\mu}{L^2\,u^2}$, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{G\,M\,\mu^2}{L^2}. \tag{8}$$

This can be recognized as a second-order, linear, inhomogeneous differential equation. The solution is a combination of the solution to the equivalent homogeneous system

$$\frac{d^2u}{d\theta^2} + u = 0 \quad\longrightarrow\quad u(\theta) = A\cos\theta \tag{9}$$

(where $A$ is a constant of integration, and we've made the choice that $u$ must be extremal at $\theta = 0, \pi$), together with the particular solution

$$u(\theta) = \frac{G\,M\,\mu^2}{L^2}. \tag{10}$$

Combining the homogeneous and particular solutions, we find

$$u(\theta) = \frac{G\,M\,\mu^2}{L^2} + A\cos\theta; \tag{11}$$

therfore, in terms of the radius,

$$r(\theta) \equiv \frac{1}{u} = \frac{L^2}{G\,M\,\mu^2(1 + e\cos\theta)}. \tag{12}$$

where we have made the association $e = AL^2/G\,M\,\mu^2$. This is the desired result.

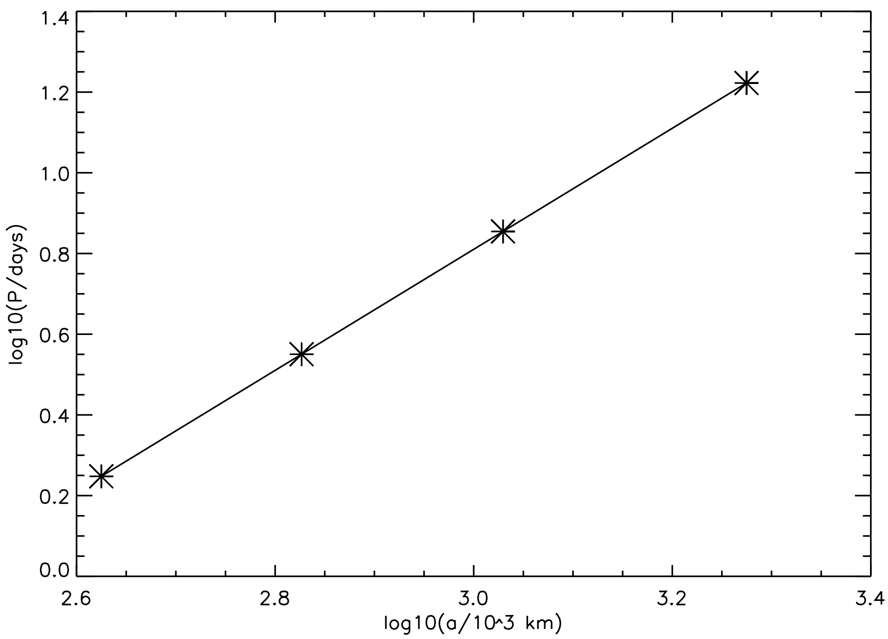


Figure 1: Graph for Q2.12