Assignment 5 — Solutions [*Revision* : 1.3]

Q10.1 The equation of hydrostatic equilibrium is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho.$$

Recalling that the definition of optical depth is

$$\mathrm{d}\tau = -\bar{\kappa}\rho\,\mathrm{d}r,$$

we can rewrite the HE equation as

$$-\bar{\kappa}\rho\frac{\mathrm{d}P}{\mathrm{d}\tau} = -g\rho.$$

Rearranging,

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{g}{\bar{\kappa}}$$

which is the desired result.

Q10.3 If the Sun is composed completely of hydrogen atoms, then the number of atoms will be given by

$$N = \frac{M_\odot}{m_\mathrm{H}}.$$

Plugging in the numbers, $N = 1.2 \times 10^{57}$.

If 10 eV is released by every atom, the total amount of energy liberated over the Sun's lifetime would be

$$E = N \cdot 10\,\mathrm{eV} = 1.9 \times 10^{39}\,\mathrm{J}$$

The lifetime of the Sun, at its present luminosity, would therefore be

$$\tau = \frac{E}{L_\odot} = 5.0 \times 10^{12}\,\mathrm{s} = 158,000\,\mathrm{yr}$$

This is much shorter than the age of the Earth (as determined from geology and paleontology), and therefore it is not possible that the Sun's energy is entirely chemical.

Q10.4 (a). The height of the Coulomb barrier between two protons is

$$U \approx \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

where r is their separation. Assuming $r \approx 10^{-15}\,\mathrm{m}$, we find $U \approx 1.44\,\mathrm{MeV} \approx 2.31 \times 10^{-13}\,\mathrm{J}$.

If this barrier can be penetrated by protons with a velocity $v = 10\,v_\mathrm{rms}$, where v_rms is the root-mean-square velocity of the Maxwell-Boltzmann distribution, then these penetrating photons must have a kinetic energy

$$E \geq 10^2 \cdot \frac{3kT}{2}.$$

(The fractional term on the right-hand side is the kinetic energy at $v = v_\mathrm{rms}$). Equating this with the barrier height derived above,

$$100\frac{3kT}{2} \geq 2.31 \times 10^{-13}\,\mathrm{J}.$$

Solving for the temperature, we obtain $T = 1.11 \times 10^8\,\mathrm{K}$. This is approaching a factor of ten larger than the actual central temperature of the Sun, $T_\mathrm{c} \approx 1.6 \times 10^7\,\mathrm{K}$.

(b). With a little algebra, eqn. (8.1) of O&C can be written in the form

$$n_v \, \mathrm{d}v = n \left(\frac{3}{2\pi v_{\mathrm{rms}}} \right)^{3/2} e^{-3v^2/2v_{\mathrm{rms}}} 4\pi \left(\frac{v}{v_{\mathrm{rms}}} \right)^2 \mathrm{d}v,$$

where $v_{\mathrm{rms}} \equiv \sqrt{3kT/m}$ is the RMS velocity. Hence, the ratio between the number of particles per unit velocity interval at $v = v_{\mathrm{rms}}$ and $v = 10\, v_{\mathrm{rms}}$ is

$$\frac{n_{10v_{\mathrm{rms}}}}{n_{v_{\mathrm{rms}}}} = \frac{100e^{-150}}{e^{-1.5}} = 3.22 \times 10^{-63}$$

(c). From the previous question, $N \approx 1.2 \times 10^{57}$. Given the ratio calculated above, we can see that, on average, not even a single proton will be moving with a velocity of ten times v_{rms}! Therefore, in the absence of tunneling there is no way that $v = 10v_{\mathrm{rms}}$ protons can account for the Sun's luminosity.

Q10.5 The pressure integral is

$$P = \frac{1}{3} \int_0^\infty m n_v v^2 \, \mathrm{d}v$$

(cf. O&C, eqn. 10.9). Substituting in the Maxwell-Boltzmann velocity distribution given by eqn. (8.1) of O&C, we have

$$P = \frac{1}{3} \int_0^\infty mn \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} 4\pi v^4 \, \mathrm{d}v.$$

Making a change of variables to $x = \sqrt{m/2kT}v$, this can be rewritten as

$$P = \frac{1}{3} mn \left(\frac{m}{2\pi kT} \right)^{3/2} 4\pi \left(\frac{2kT}{m} \right)^{5/2} \int_0^\infty x^4 e^{-x^2} \, \mathrm{d}x = \frac{8}{3\sqrt{\pi}} nkT \int_0^\infty x^4 e^{-x^2} \, \mathrm{d}x$$

The definite integral evaluates (e.g., using Mathematica) to $3\sqrt{\pi}/8$. Hence, we obtain

$$P = nkT,$$

which is the ideal gas law.

Q10.17 To prove that the $n = 0$ polytrope has the solution

$$D_0(\xi) = 1 - \frac{\xi^2}{6},$$

we can substitute it into the Lane-Emden equation:

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\xi^2 \frac{\mathrm{d}D_n}{\mathrm{d}\xi} \right] = -D_n^n.$$

The left-hand side evaluates to

$$\frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\xi^2 \frac{\mathrm{d}D_n}{\mathrm{d}\xi} \right] = \frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\xi^2 \cdot -\frac{\xi}{3} \right] = -1$$

The right-hand side trivially evaluates to -1 also (since $x^0 = 1$ for any $x \neq 0$). Hence, the solution above satisfies the Lane-Emden equation.

The remaining step is to verify that the solution also satisfies the boundary conditions. At $\xi = 0$, we have

$$D_0(0) = 1$$

and

$$D_0'(0) = 0,$$

which does indeed satisfy the conditions $D_n(0) = 1$ and $D_n'(0) = 0$. Moreover, $D_0(\xi) = 0$ at $\xi = \xi_1 = \sqrt{6}$.

Q10.18 The density of a polytrope is given by

$$\rho = \rho_{\rm c} D_n^n$$

(cf. p. 336 of O&C), where $\rho_{\rm c}$ is the central density and D_n the appropriate solution of the Lane-Emden equation. With $n = 0$, this becomes

$$\rho = \rho_{\rm c},$$

and so the $n = 0$ polytrope can be recognized as having a uniform density structure. This polytrope is often referred to as the 'homogeneous, incompressible' case.

Q10.20 (a). See Fig. 1 for the graph.

(b). The curves indicate that the density is more centrally concentrated for increasing polytropic index. This statement can be placed on a more-rigorous footing by noting that the mean density of a polytrope is

$$\bar{\rho} = \frac{3M}{4\pi R^3} = -3\rho_{\rm c}\xi_1^{-1} \left.\frac{{\rm d}D_n}{{\rm d}\xi}\right|_{\xi_1}$$

(using expressions given for the mass and radius by O&C). Hence, the ratio between the central density and $\bar{\rho}$ (which measures the degree of central concentration) is

$$\frac{\rho_{\rm c}}{\bar{\rho}} = \frac{1}{3}\left[\xi_1^{-1} \cdot - \left.\frac{{\rm d}D_n}{{\rm d}\xi}\right|_{\xi_1}\right]^{-1}$$

Consulting any table of polytropes (or using *Poly-Web*) will show that both terms in the brackets are monotonically decreasing functions of n; thus, the degree of central concentration will be a monotonically *increasing* function of n.

(c). An adiabatically convective model has $n = 1.5$, whereas a model in radiative equilibrium has $n = 3$. Therefore, the latter should be more centrally concentrated than the former.

(d). This question is too vague to answer.

***EZ-Web* Polytrope Question** If the Sun's structure is described by a polytrope, then the pressure and density should be related by

$$P = K\rho^{(n+1)/n}$$

for fixed K and n. Taking the logarithm of both sides gives

$$\log_{10} P = \frac{n+1}{n}\log_{10}\rho + \log_{10} K;$$

thus, a plot of the solar structure in the $\log_{10} P - \log_{10}\rho$ plane can be used to determine the polytropic index.

Fig. 2 shows such a plot, for an *EZ-Web* model of the present-day Sun. The best fit line to the pressure/density data has a slope 1.41; hence, the polytropic index for the Sun is estimated from $(n+1)/n \approx 1.41$, as $n \approx 2.44$.

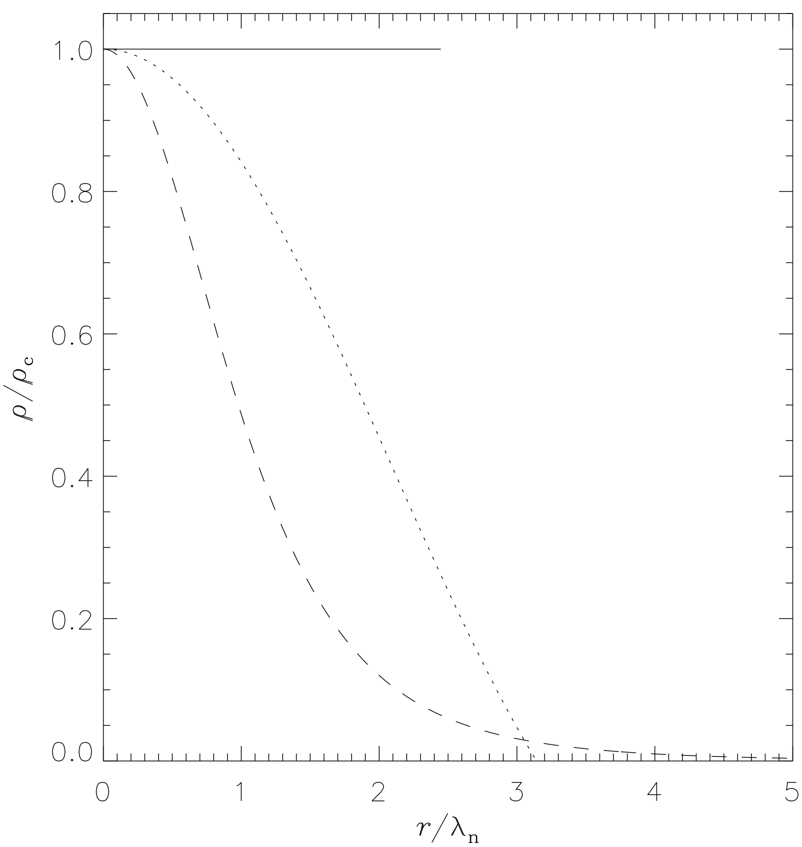

Figure 1: The scaled density ρ/ρ_c plotted as a function of scaled radius r/λ_n $(=\xi)$ for polytropes having indices $n = 0$ (solid), $n = 1$ (dotted) and $n = 5$ (dashed). Note that the $n = 0$ curve terminates at $r/\lambda_n = \sqrt{6}$, because this point corresponds to the stellar surface.

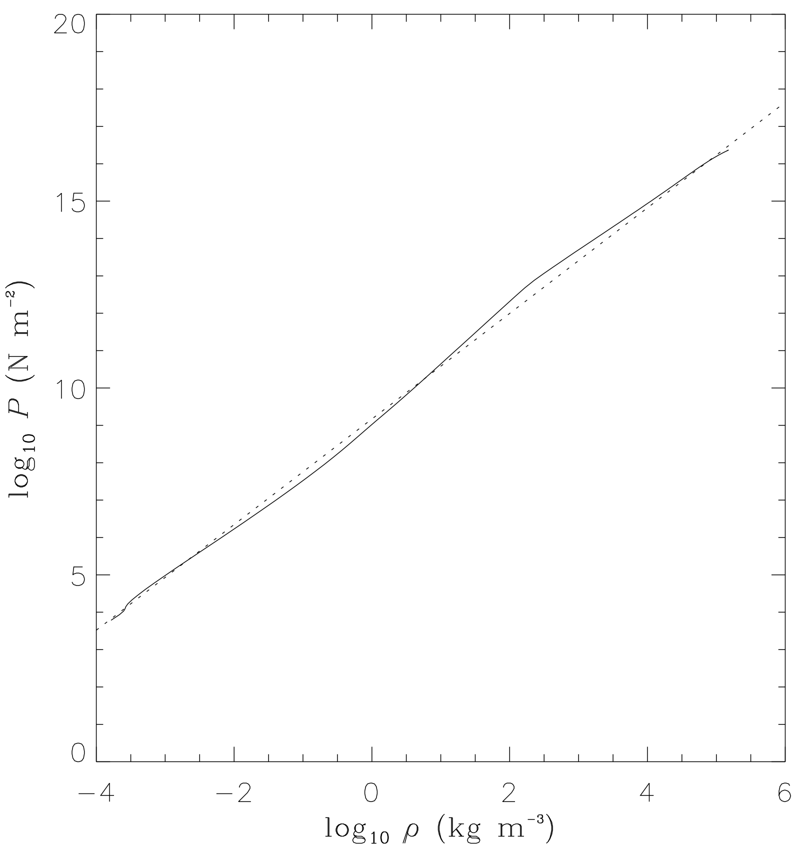

Figure 2: An *EZ-Web* model for the present-day Sun, plotted in the logarithmic pressure-density plane (solid). The dotted line shows the best fit to the *EZ-Web* data; it has a slope 1.41 and an intercept 9.17.