Assignment 4 — Solutions [*Revision* : 1.4]

Q9.7 We typically see a optical distance $\tau \approx 2/3$ through an opaque medium. Using

$$\tau = \kappa \rho s,$$

for constant $\kappa = 0.03\,\mathrm{m^2kg^{-1}}$ and $\rho = 1.2\,\mathrm{kgm^{-3}}$, gives a physical distance $s = 18.5\,\mathrm{m}$ — not very far at all.

Q9.10 Measuring the slope of the straight part of the $\log \rho = -1$ curve[1] in Fig. 9.10 of Ostlie & Carroll gives $\mathrm{d}\log\kappa/\mathrm{d}\log T \approx -3.5/1$. This corresponds to a power law $\kappa \propto T^{-3.5}$, which agrees with Kramers' law.

Q9.18 Within the Eddington approximation, a grey, plane-parallel atmosphere obeys the relations

$$\langle I \rangle = \frac{1}{2}(I_{\mathrm{out}} + I_{\mathrm{in}})$$

and

$$F_{\mathrm{rad}} = \pi(I_{\mathrm{out}} - I_{\mathrm{in}})$$

(see eqns. 9.46 and 9.47 of O&C). These two equations can be combined to give the outward and inward specific intensities as

$$I_{\mathrm{out}} = \langle I \rangle + \frac{F_{\mathrm{rad}}}{2\pi}$$

and

$$I_{\mathrm{in}} = \langle I \rangle - \frac{F_{\mathrm{rad}}}{2\pi}.$$

Now, in radiative equilibrium, the mean intensity varies with vertical optical depth as

$$\langle I \rangle = \frac{3F_{\mathrm{rad}}}{4\pi}\left(\tau_{\mathrm{v}} + \frac{2}{3}\right)$$

(see eqn. 9.50 of O&C). Hence, the outward and inward intensities as a function of τ_{v} are

$$I_{\mathrm{out}} = \frac{3F_{\mathrm{rad}}}{4\pi}\left(\tau_{\mathrm{v}} + 4/3\right)$$

and

$$I_{\mathrm{in}} = \frac{3F_{\mathrm{rad}}}{4\pi}\left(\tau_{\mathrm{v}}\right).$$

The ratio of these is

$$\frac{I_{\mathrm{out}}}{I_{\mathrm{in}}} = 1 + \frac{4}{3\tau_{\mathrm{v}}};$$

this is within 1% of being isotropic (i.e., $I_{\mathrm{out}}/I_{\mathrm{in}} = 1$) when $\tau_{\mathrm{v}} = 133.3$.

Q9.19 From eqn. 9.53 of O&C, the temperature structure of an LTE, plane-parallel, grey, Eddington approximation atmosphere is given by

$$T^4 = \frac{3}{4}T_{\mathrm{e}}^4\left(\tau_{\mathrm{v}} + \frac{2}{3}\right).$$

At the top of the atmosphere, $\tau_{\mathrm{v}} = 0$ and hence

$$T_{\mathrm{top}} = 2^{-1/4}T_{\mathrm{e}} \approx 0.84T_{\mathrm{e}}.$$

For an effective temperature $T_{\mathrm{e}} = 5777\,\mathrm{K}$ (the value for the Sun), the top of the atmosphere has a temperature $T_{\mathrm{top}} = 4858\,\mathrm{K}$.

[1] This is a good choice of curve to measure, since it has an extended straight part.

Q9.21 The formal solution of the radiative transfer equation, as given in eqn. 9.54 of O&C, is

$$I_\lambda(0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} - \int_{\tau_{\lambda,0}}^{0} S_\lambda e^{-\tau_\lambda}\, \mathrm{d}\tau_\lambda.$$

If no radiation enters from outside, then $I_{\lambda,0} = 0$. Moreover, if the source function S_λ does not vary with position, then it can be brought out in front of the integral:

$$I_\lambda(0) = -S_\lambda \int_{\tau_{\lambda,0}}^{0} e^{-\tau_\lambda}\, \mathrm{d}\tau_\lambda = S_\lambda \left(1 - e^{-\tau_{\lambda,0}}\right).$$

1. When $\tau_{\lambda,0} \gg 1$, $e^{\tau_{\lambda,0}} \to 0$, and thus the emergent intensity becomes

   $$I_\lambda(0) = S_\lambda.$$

   If the gas is in thermodynamic equilibrium, then $S_\lambda = B_\lambda$, and so the emergent intensity is a blackbody spectrum:

   $$I_\lambda(0) = B_\lambda.$$

2. When $\tau_{\lambda,0} \ll 1$, the exponential in the formal solution can be expanded as $\exp^{-\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0}$, and thus the emergent intensity becomes

   $$I_\lambda(0) = S_\lambda \tau_{\lambda,0}.$$

   Writing optical depth in terms of the opacity, this becomes

   $$I_\lambda(0) = S_\lambda \int_0^L \kappa_\lambda \rho\, \mathrm{d}s = \int_0^L S_\lambda \kappa_\lambda \rho\, \mathrm{d}s$$

   where the second expression has taken the constant source function back inside the integral. Writing the source function in terms of the emissivity and opacity, this then becomes

   $$I_\lambda(0) = \int_0^L \frac{j_\lambda}{\kappa_\lambda} \kappa_\lambda \rho\, \mathrm{d}s = \int_0^L j_\lambda \rho\, \mathrm{d}s.$$

   This expression indicates that the spectrum will exhibit high intensities over wavelengths where the emissivity is large — i.e., emission lines.

Q9.22 This question is quite similar to the previous one, but it is assumed that radiation with intensity $I_{\lambda,0}$ enters the bottom of the slab; therefore, the constant-source function solution is

$$I_\lambda(0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} + S_\lambda \left(1 - e^{-\tau_{\lambda,0}}\right).$$

1. When $\tau_{\lambda,0} \gg 1$, $e^{\tau_{\lambda,0}} \to 0$, and thus the emergent intensity becomes

   $$I_\lambda(0) = S_\lambda.$$

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   $$I_\lambda(0) = B_\lambda.$$

2. When $\tau_{\lambda,0} \ll 1$, the exponential in the formal solution can be expanded as $\exp^{-\tau_{\lambda,0}} \approx 1 - \tau_{\lambda,0}$, and thus the emergent intensity becomes

   $$I_\lambda(0) = I_{\lambda,0}(1 - \tau_{\lambda,0}) + S_\lambda \tau_{\lambda,0} = I_{\lambda,0} \left[1 - \tau_{\lambda,0} \left(1 - \frac{S_\lambda}{I_{\lambda,0}}\right)\right]$$

   If $I_{\lambda,0} > S_\lambda$, then $I_\lambda(0) < I_{\lambda,0}$ (because the term in parentheses is positive) and we see absorption lines superimposed on the spectrum of the incident radiation. Conversely, if $I_{\lambda,0} < S_\lambda$, then $I_\lambda(0) > I_{\lambda,0}$ and we see emission lines superimposed on the spectrum.

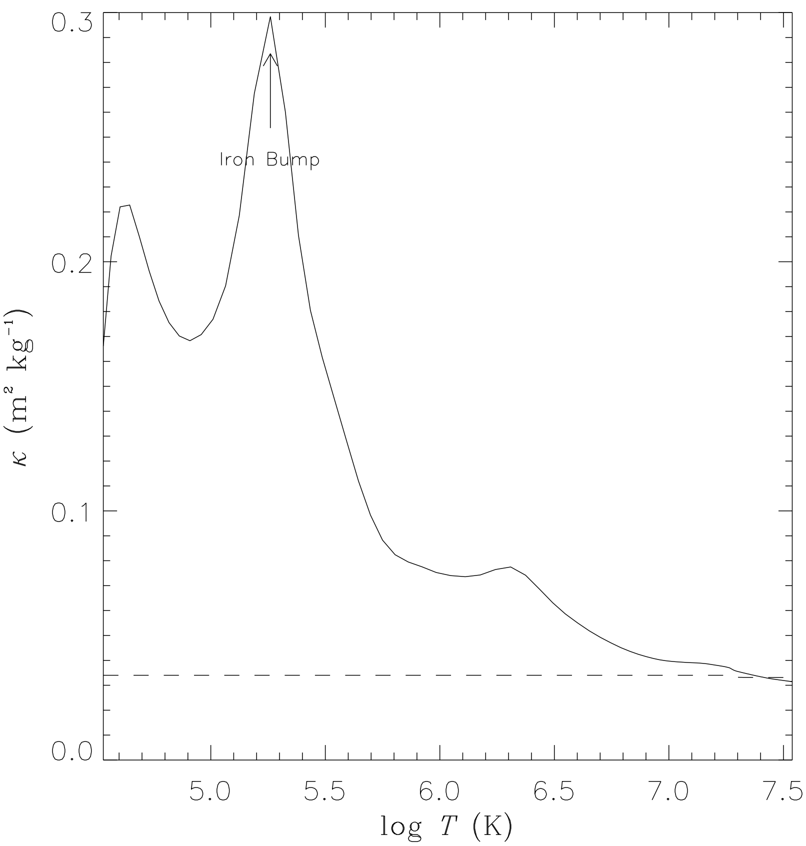

Figure 1: The Rosseland mean opacity plotted as a function of $\log T$ for the $20\,M_{\odot}$ *EZ-Web* model. The arrow indicates the location of the iron bump, and the dashed line shows the opacity calculated from the Thomson-scattering formula. Note how the two curves cross at the right-hand end of the plot.

EZ-Web $20\,M_{\odot}$ **Question** See Fig. 1 for the opacity plot, and Fig. 2 for the photon energy plot. The latter plot confirms that in the very center of the $20\,M_{\odot}$ model the photon energy is non-negligible compared to the electron rest-mass energy, and departures from the Thomson-scattering formula are to be expected. These departures are associated with the recoil of the electron required by relativistic momentum conservation. In effect, the scattering has become inelastic, and we speak of Compton scattering rather than Thomson scattering.

EZ-Web **ZAMS Question** See Fig. 3 for the theoretical HR diagram, and Figs. 4 and 5 for the mass-luminosity and mass-radius plots. Straight-line fits to these plots give the following power-law relations:

$$L/L_{\odot} \approx (M/M_{\odot})^{3.48}$$

and

$$R/R_{\odot} \approx (M/M_{\odot})^{0.68}.$$

If 0.1% of a star's rest mass is converted into energy over its lifetime, then the lifetime must be

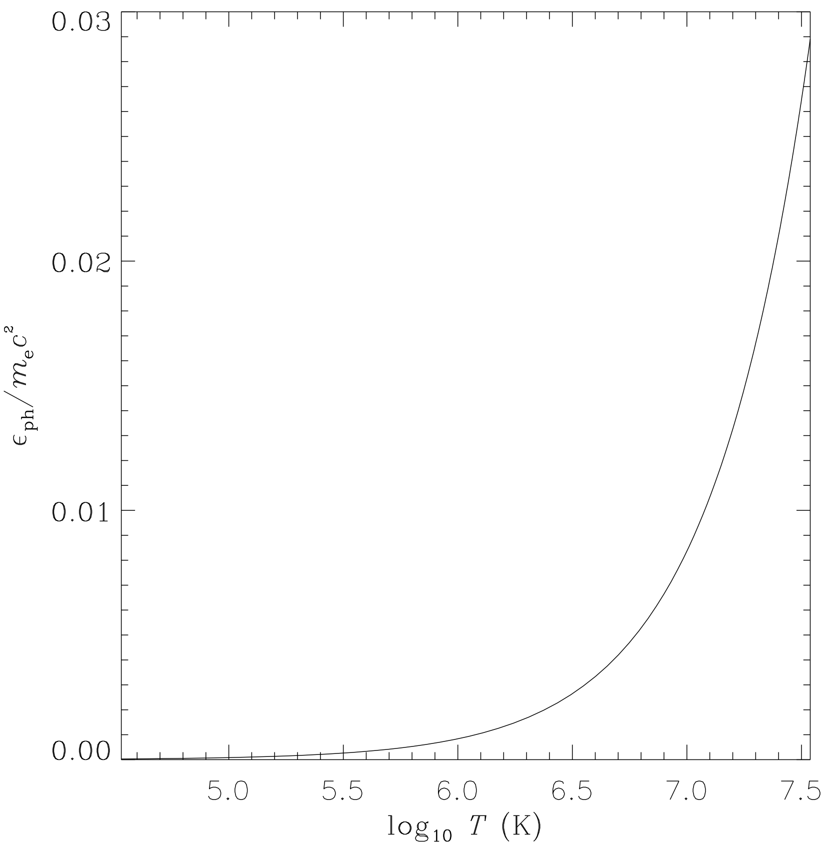

Figure 2: The ratio between Wien-peak photon energy and electron rest-mass energy, plotted as a function of $\log T$ for the $20\,M_\odot$ *EZ-Web* model. This ratio becomes non-negligible in the center of the star, indicating the breakdown of the Thomson-scattering formula.

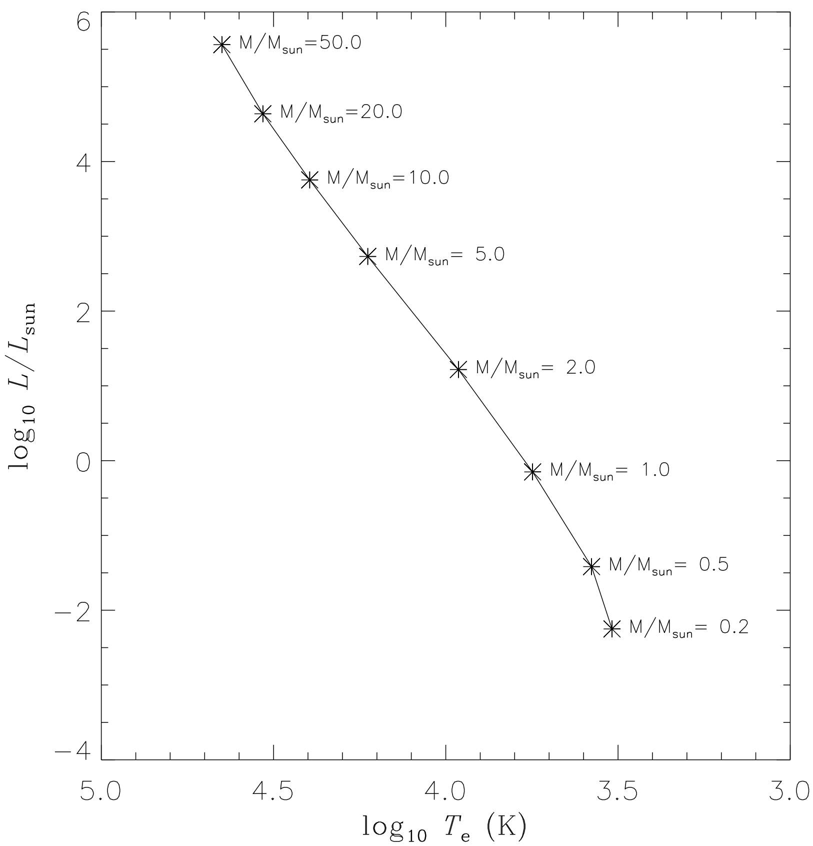

Figure 3: An HR diagram showing the ZAMS models (asterisks) calculated using *EZ-Web*.

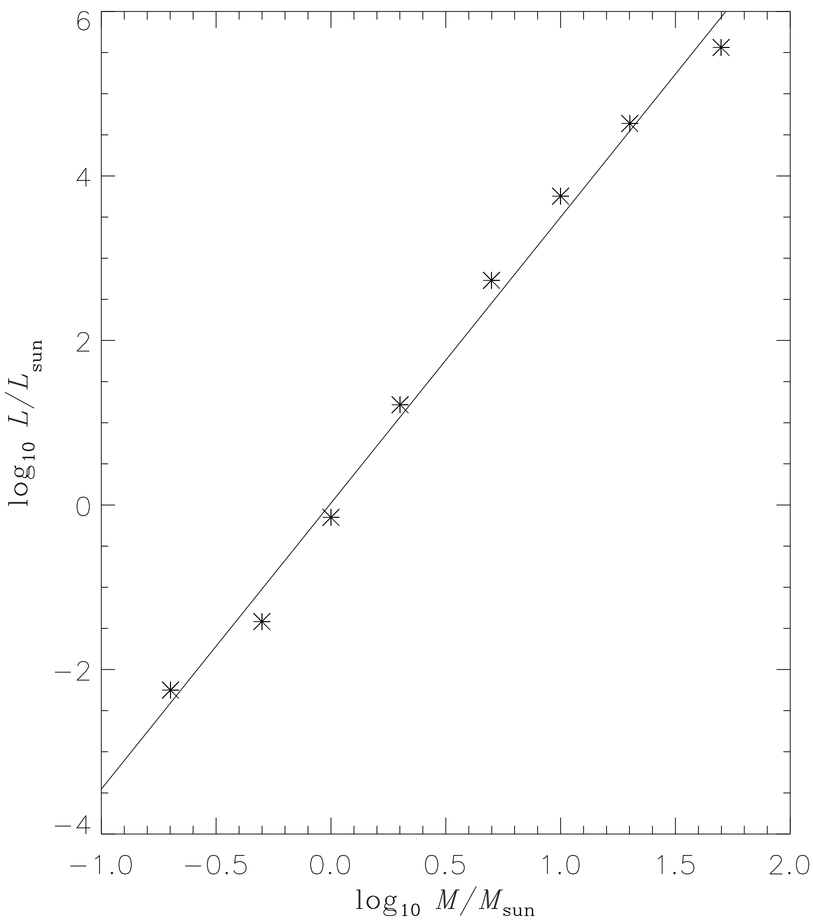

Figure 4: The mass-luminosity relation for the *EZ-Web* ZAMS models (asterisks). The straight line shows the best fit to the data; its slope is 3.48 and its intercept is 0.022.

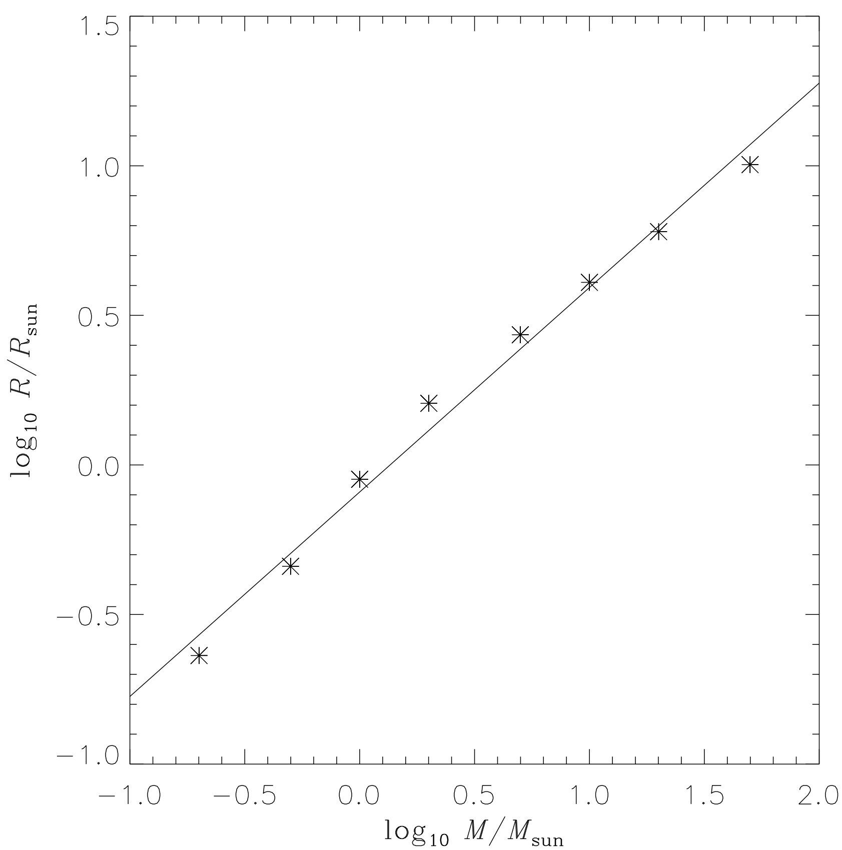

Figure 5: The mass-radius relation for the *EZ-Web* ZAMS models (asterisks). The straight line shows the best fit to the data; its slope is 0.68 and its intercept is -0.094.

equal to the total energy converted divided by the energy conversion rate (i.e., the luminosity):

$$\tau \approx \frac{0.001 M c^2}{L} \approx \frac{0.001 c^2 M_\odot}{L_\odot} \frac{M/M_\odot}{L/L_\odot}$$

Using the derived mass-luminosity relation, we therefore find that

$$\tau \approx \frac{0.001 c^2 L_\odot}{M_\odot} (M/M_\odot)^{-2.48} \approx 1.5 \times 10^{10}\,\mathrm{yr} \cdot (M/M_\odot)^{-2.48}.$$

For a $50\,M_\odot$ star, this expression gives a lifetime $\tau \approx 9 \times 10^5\,\mathrm{yr}$, which is a factor of $50^{-2.48} = 6 \times 10^{-5}$ shorter than the lifetime of the Sun (as calculated using the same expression).

To derive a mass-effective temperature relation, we make use of the equation defining effective temperature,

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4.$$

Using the fitted relations to eliminate the luminosity and radius, we obtain

$$L_\odot (M/M_\odot)^{3.48} \approx 4\pi R_\odot^2 (M/M_\odot)^{1.36} \sigma T_{\mathrm{eff}}^4.$$

Solving for the effective temperature,

$$T_{\mathrm{eff}} \approx \left(\frac{L_\odot}{4\pi R_\odot^2 \sigma} (M/M_\odot)^{2.12} \right)^{1/4} \approx 5780\,\mathrm{K} \cdot (M/M_\odot)^{0.53}.$$

Alternatively, this can be written as

$$M/M_\odot \approx (T_{\mathrm{eff}}/5780\,\mathrm{K})^{1.89}.$$

Applying this to the specified main sequence stars, we find:

1. σ Ori A : $T_{\mathrm{eff}} = 32,000\,\mathrm{K}, M \approx 25\,M_\odot$
2. Regulus : $T_{\mathrm{eff}} = 10,300\,\mathrm{K}, M \approx 3.0\,M_\odot$
3. Procyon : $T_{\mathrm{eff}} = 6,650\,\mathrm{K}, M \approx 1.3\,M_\odot$
4. ϵ Indii : $T_{\mathrm{eff}} = 4,280\,\mathrm{K}, M \approx 0.57\,M_\odot$