Assignment 3 — Solutions [$Revision$: 1.1]

Q8.5 For neutral hydrogen, the population ratio of the $n = 1$ and $n = 2$ energy levels can be obtained from the Boltzmann equation,

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT}.$$

This may be rewritten in terms of the temperature, as

$$T = \frac{1}{k} \frac{-(E_2 - E_1)}{\ln(N_2/N_1) - \ln(g_2/g_1)}.$$

The statistical weight is given by $g_n = 2n^2$, and $E_2 - E_1 = 10.2\,\mathrm{eV} = 1.63 \times 10^{-18}\,\mathrm{J}$. Thus at 1% excitation (i.e., $N_2/N_1 = 0.01$) $T = 19,800\,\mathrm{K}$, while at 10% excitation $T = 32,100\,\mathrm{K}$.

Q8.8 Equation (8.7) for the partition function is

$$Z = \sum_{j=1}^{\infty} g_j e^{-(E_j-E_1)/kT}.$$

For hydrogen ($g_j = 2j^2$) at any $T > 0$, this sum formally diverges. However, high-n levels whose effective electron orbital radius is larger than the typical inter-atom separation cannot be stably bound, and must be ignored in the sum. For these levels, the energy uncertainty ΔE ($\approx \hbar/\tau$, where τ is the level lifetime) is larger than the level's energy E, and the level merges with the continuum energy states that correspond to an ionized system.

Q8.10 (a). Eqn. (8.9) is the Saha equation,

$$\frac{N_{i+1}}{N_i} = \frac{2kT}{P_\mathrm{e}} \frac{Z_{i+1}}{Z_i} \left(\frac{2\pi m_\mathrm{e} kT}{h^2}\right)^{3/2} e^{-\chi_i/kT},$$

written in terms of the electron pressure P_e rather than the electron number density n_e. Plugging in the supplied values, we obtain the following results:

(i) At 5,000 K, $N_\mathrm{II}/N_\mathrm{I} = 1.89 \times 10^{-18}$ and $N_\mathrm{III}/N_\mathrm{II} = 4.32 \times 10^{-49}$.

(ii) At 15,000 K, $N_\mathrm{II}/N_\mathrm{I} = 0.997$ and $N_\mathrm{III}/N_\mathrm{II} = 2.42 \times 10^{-11}$.

(iii) At 25,000 K, $N_\mathrm{II}/N_\mathrm{I} = 7.24 \times 10^{3}$ and $N_\mathrm{III}/N_\mathrm{II} = 1.78 \times 10^{-3}$.

(b).

$$N_\mathrm{II}/N_\mathrm{total} = \frac{N_\mathrm{II}}{N_\mathrm{I} + N_\mathrm{II} + N_\mathrm{III}} = \frac{1}{N_\mathrm{I}/N_\mathrm{II} + 1 + N_\mathrm{III}/N_\mathrm{II}} = \frac{1}{1 + (N_\mathrm{II}/N_\mathrm{I})^{-1} + (N_\mathrm{III}/N_\mathrm{II})}$$

(c). See Fig. 1

Q8.14 This is due to the apperance of the electron number density n_e (or the electron pressure P_e) in the denominator of the Saha equation. The number density is lower in giant stars than main sequence stars, due to the more-diffuse atmospheres of the former. Therefore, to produce the same ionization state (and hence ostensibly the same spectral type), the atmospheric temperature of a giant star must be lower than for a main-sequence star by an amount that compensates for the reduced n_e.

Q8.16 Locating Fomalhaut on the HR diagram in Fig. 8.16 gives its spectral type as around A3, and its absolute visual magnitude as $M_V \approx 2.5$. Thus, the distance modulus is $V - M_V = -1.3$, and the distance to the star is calculated as $d = 10^{(V-M_V)/5} \times 10\,\mathrm{pc} = 5.5\,\mathrm{pc}$. This compares against the distance $d = 7.69\,\mathrm{pc}$ measured by HIPPARCOS.

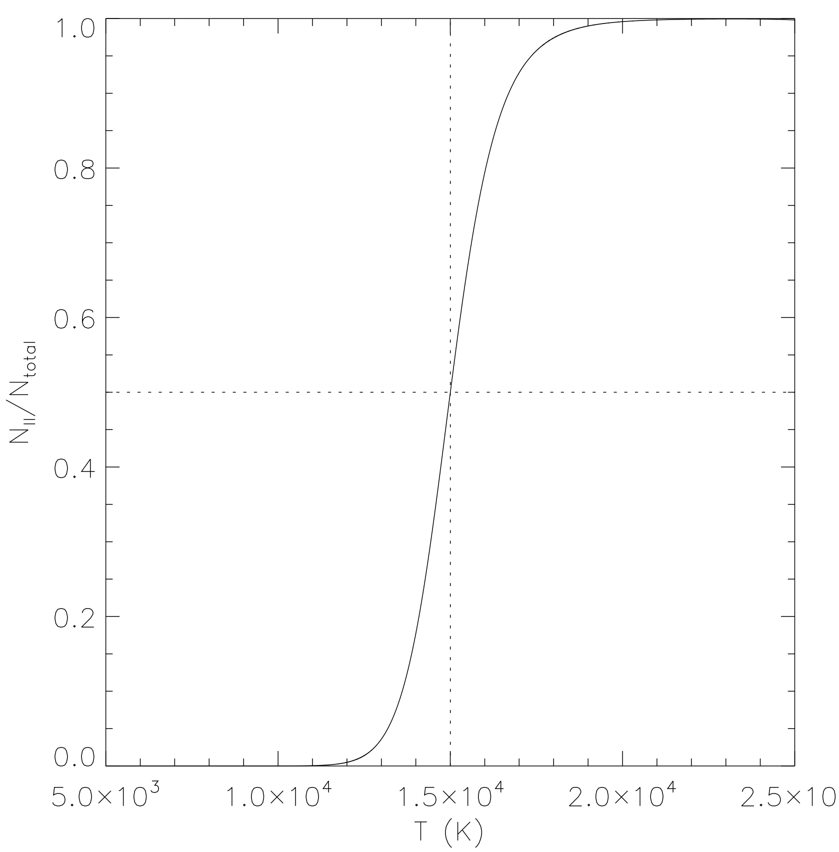

Figure 1: The fraction of helium atoms in the He II ionization state, plotted as a function of temperature. The dotted lines indicate the middle of the ionization zone ($T \approx 15,000\,\mathrm{K}$), where $N_{\mathrm{II}}/N_{\mathrm{total}} = 0.5$.

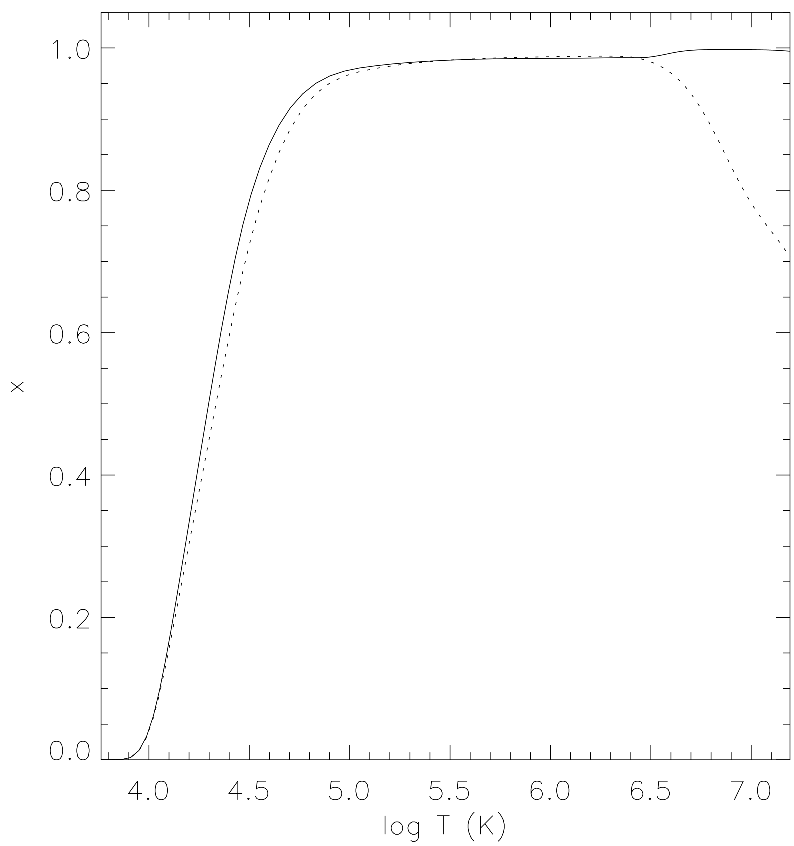

Figure 2: The ionization fraction of hydrogen in an *EZ-Web* model for the present-day Sun, plotted as a function of temperature. The solid line shows the values calculated from the X and X^+ data, while the dotted line shows the results obtained using the Saha equation.

EZ-Web The ionization fraction graph is shown in Fig. 2. The values predicted by the Saha equation begin to drop toward the center, because the electron density becomes very high (recall that n_{e} appears in the denominator of the Saha equation). However, this drop does not happen in reality (witness the solid line in the figure), because at high densities the electrostatic potentials of neighboring nucleii overlap, lowering the atoms' ionization potentials and thus maintaining a state of complete ionization.

Bonus The radius and luminosity of the model are $R = 1.08\,R_\odot$ and $L = 4.32\,L_\odot$, indicating an effective temperature $T_{\mathrm{eff}} = (L/4\pi R^2\sigma)^{1/4} = 8020\,\mathrm{K}$. This corresponds to a Wien peak of $\lambda_{\mathrm{max}} = 0.0029 m/T_{\mathrm{eff}} = 361\,\mathrm{nm}$. This is in the near-UV, indicating that the early Universe — which was populated by metal-poor stars, since the metals had not yet been created via supernovae — would have been rich in UV radiation.