Assignment 2 — Solutions [$Revision$: 1.3]

Q2.8 (a). The general form of Kepler's third law is

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3.$$

Assuming that the mass of Hubble is negligible compared to that of the Earth, this can be approximeted by

$$P^2 \approx \frac{4\pi^2}{GM_\oplus} a^3$$

Plugging in the orbital radius $a = R_\odot + 610\,\mathrm{km} = 6990\,\mathrm{km}$ and the Earth's mass $M_\oplus = 5.96 \times 10^{24}\,\mathrm{kg}$ gives an orbital period of $P = 5820\,\mathrm{s} = 97\,\mathrm{min}$. (Observing time on Hubble is always scheduled in integer multiples of this 97-minute orbital period.)

(b). In a geosynchronous orbit, the orbital period is exactly equal to one day. Using the approximate form of Kepler's third law above, for $P = 1\,\mathrm{d}$ the orbital radius is $a = 4.22 \times 10^7\,\mathrm{m}$. Thus, the altitude of the orbit is $a - R_\oplus = 3.58 \times 10^7\,\mathrm{m} = 35,800\,\mathrm{km}$.

(c). No, it is not possible. Only geosynchronous orbits that lie in the Earth's equatorial plane will remain parked over a specific location; geosynchronous orbits in other planes (which *must* be centered on the Earth) will remain at the same terrestrial longitude, but their latitude will wander north and south over one day.

Q2.12 (a). See Fig. 1

(b). The slope of the best-fit line is $1.50 = 3/2$.

(c). Assuming that the masses of the Galilean moons are negligible, the general form of Kepler's third law applied to the Jupiter system is

$$P^2 \approx \frac{4\pi^2}{GM_{♃}} a^3.$$

Taking the logarithm of both sides,

$$2 \log P \approx \log 4 + 2 \log \pi - \log G - \log M_{♃} + 3 \log a.$$

Rearranging,

$$\log P \approx \frac{1}{2} \left(\log 4 + 2 \log \pi - \log G - \log M_{♃} \right) + \frac{3}{2} \log a.$$

The first quantity on the right-hand side must equal the intercept of the $\log P$ vs $\log a$ line, which is -7.75. Solving for Jupiter's mass then gives $M_{♃} = 1.88 \times 10^{27}\,\mathrm{kg}$.

Q2.14 (a). For any general system, the period and eccentricity are insufficient to calculate the semi-major axis. *However*, in the special case of the solar-system, we can use Kepler's third law in its original form:

$$(P/\mathrm{yr})^2 = (a/\mathrm{au})^3.$$

Plugging in the period of 76 yr gives a semi-major axis of $a = 17.9\,\mathrm{AU} = 2.68 \times 10^{12}\,\mathrm{m}$.

(b). Neglecting the masses of the planets, comets and asteroids, the general form of Kepler's third law for the solar system is

$$P^2 = \frac{4\pi^2}{GM_\odot} a^3$$

Using the values of P and a from Halley's comet, the mass of the Sun is calculated as $M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$.

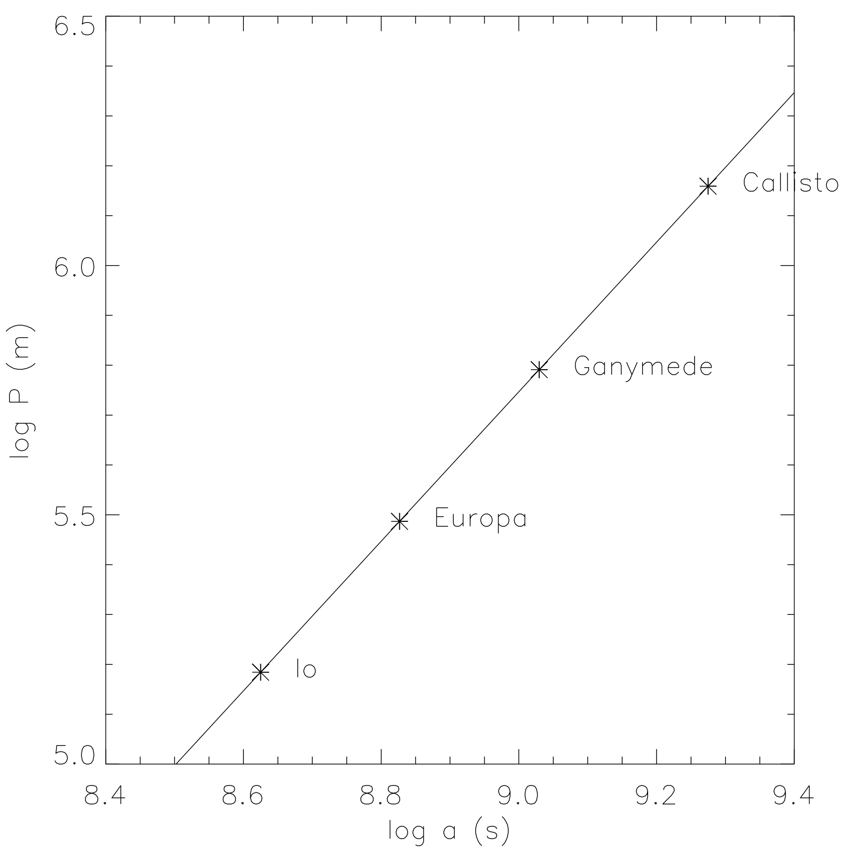

Figure 1: $\log P$ vs. $\log a$ for the Galilean moons. The solid line is the least-squares best-fit line; it has a slope of 1.50 and an intercept of -7.75.

(c). At perhelion $r_{\mathrm{p}} = (1-e)a = 8.78\times10^{10}$ m, while at aphelion $r_{\mathrm{a}} = (1+e)a = 5.28\times10^{12}$ m. Note that we have neglected the fact that the Sun isn't at the center-of-mass of the solar system; this turns out to be only a 1% correction.

(d). The orbital speed can be calculated from conservation of energy. The total energy of the comet is

$$E = \frac{1}{2}mv^2 - \frac{GM_\odot m}{r}$$

(kinetic plus potential). From equation 2.35 of Ostlie & Carroll, the total energy is also given by

$$E = -\frac{GM_\odot m}{2a}$$

Equating the two values and dividing through by the comet mass m,

$$\frac{1}{2}v^2 - \frac{GM_\odot}{r} = -\frac{GM_\odot}{2a}.$$

Solving for the speed,

$$v = \sqrt{GM_\odot\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Plugging in the above values for the semi-major axis and perihelion/aphelion radii, we find that $v_{\mathrm{p}} = 5.45\times10^4\,\mathrm{ms^{-1}} = 54.5\,\mathrm{kms^{-1}}$ and $v_{\mathrm{a}} = 9.07\times10^2\,\mathrm{ms^{-1}} = 0.907\,\mathrm{kms^{-1}}$. On the semi-minor axis, the radius is $r_{\mathrm{s}} = a = 2.68\times10^{12}$ m, and so $v_{\mathrm{s}} = 7.04\times10^3\,\mathrm{ms^{-1}} = 7.04\,\mathrm{kms^{-1}}$.

(e). The ratio of kinetic energies is $v_{\mathrm{p}}^2/v_{\mathrm{a}}^2 = 3610$.

Q5.1 (a). The radial velocity can be determined from the Doppler shift formula:

$$\frac{v_r}{c} = \frac{\Delta\lambda}{\lambda}$$

Measured in air, the rest wavelength of Hα is 656.281 nm; hence, for Barnard's star $\Delta\lambda = 656.034 - 656.281 = -0.247$ nm. Solving for the radial velocity, $v_r = -1.13\times10^5\,\mathrm{ms^{-1}}$ (the negative sign indicates that the star is approaching us).

(b). The transverse velocity is given by

$$v_\theta = \mu d$$

where μ is the proper motion and d is the distance. With $\mu = 10.3577''\mathrm{yr^{-1}}$ and $d = 1/0.54901'' = 1.82$ pc, we find that $v_\theta = 18.9\,\mathrm{AUyr^{-1}}$. In standard units, this is $v_\theta = 8.94\times10^4\,\mathrm{ms^{-1}}$.

(c). The speed is given by

$$v = \sqrt{v_r^2 + v_\theta^2};$$

plugging in the numbers, we find $v = 1.44\times10^5\,\mathrm{ms^{-1}} = 144\,\mathrm{kms^{-1}}$.

Q5.2 (a). The diffraction angle for light with wavelength λ is found from

$$d\sin\theta = n\lambda$$

(see p. 113 of Ostlie & Carroll). Taking the implicit derivative of both sides gives

$$d\cos\theta\Delta\theta = n\Delta\lambda,$$

which relates a small change in wavelength $\Delta\lambda$ to the corresponding small change in diffration angle. For a grating with 300 lines per millimeter, $d = 300^{-1}\,\mathrm{mm} = 3.33\times$

10^{-6} m, and the second-order ($n = 2$) spectrum will have the sodium D lines at a diffraction angle $\theta = 0.361\,\text{rad} = 21.7°$. (This comes from solving the first equation above for θ, using the average wavelength $\lambda = 589\,\text{nm}$ of the two D lines). The angular separation of the two lines, which have $\Delta\lambda = 0.597\,\text{nm}$, is then found from the second equation as $\Delta\theta = 3.83 \times 10^{-4}\,\text{rad} = 0.0219°$.

(b). The smallest wavelength that a diffraction grating can resolve is

$$\Delta\lambda = \frac{\lambda}{nN}.$$

With $\lambda = 589\,\text{nm}$ and $\Delta\lambda = 0.597\,\text{nm}$ and $n = 2$, the number of illuminated grating lines when the sodium D lines are just resolved is found as $N = 494$.

Q5.11 The wavelengths of hydrogen emission follow Rydberg's formula,

$$\frac{1}{\lambda} = R_H \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right),$$

where n_1 and n_2 are the principal quantum numbers of the lower (final) and upper (initial) levels involved in the electron transition. For a given n_1, the shortest-wavelength emission is given by the limit $n_2 \to \infty$:

$$\frac{1}{\lambda_{\text{lim}}} = \frac{R_H}{n_1^2},$$

or

$$\lambda_{\text{lim}} = \frac{n_1^2}{R_H}$$

With the Rydberg constant $R_H = 1.097 \times 10^7\,\text{m}^{-1}$, the series limits are 91.2 nm ($n_1 = 1$; Lyman series), 365 nm ($n_1 = 2$; Balmer series) and 821 nm ($n_1 = 3$; Paschen series). These limits fall in the far-UV, near-UV and near-IR parts of the electromagnetic spectrum.

Q5.15 (a). If the electron spends 10^{-8} s in the first excited state ($n = 2$), then the energy of this state is uncertain by an amount given by

$$\Delta E_2 \approx \frac{\hbar}{\Delta t_2} \approx \frac{\hbar}{10^{-8}\,\text{s}} \approx 1.05 \times 10^{-26}\,\text{J} \approx 1.05 \times 10^{-26}\,\text{eV}$$

(b). The wavelength of the photon involved in a transition between $n = 1$ and $n = 2$ is given by

$$\frac{hc}{\lambda} = E_2 - E_1.$$

Taking the implict derivative of both sides,

$$-\frac{hc\Delta\lambda}{\lambda^2} = \Delta E_2 - \Delta E_1.$$

We can neglect the uncertainty ΔE_1 in the ground-state energy, since we assume that the lifetime in this (stable) state is arbitrarily long. Hence,

$$\Delta\lambda = \frac{\lambda^2}{hc}\Delta E_2.$$

(I've dropped the minus sign, since the signs of $\Delta\lambda$ and ΔE_2 are unimportant). Plugging in the numbers, with $\lambda = 122\,\text{nm}$ for the Lyman α transition, gives $\Delta\lambda = 7.8 \times 10^{-7}\,\text{nm}$ — a tiny amount of broadening!

Q7.4 (a). The generalized form of Kepler's third law is

$$P^2 = \frac{4\pi^2}{G(m_A + m_B)} a^3$$

The period is given as $P = 49.94\,\mathrm{yr}$, while the semi-major axis of the reduced mass can be calculated from the parallax and angular extent

$$a = 7.61(d/\mathrm{pc})\,\mathrm{AU} = 7.61(1/0.37921) = 20.07\,\mathrm{AU} = 3.00 \times 10^{12}\,\mathrm{m}$$

(note how easily everything is done using arseconds, parsecs and AU). Kepler's third law then gives the total mass $M = m_A + m_B = 6.43 \times 10^{30}\,\mathrm{kg}$.

For the individual components, the semi-major axes are given by $a_A = m_B/M$ and $a_B = m_A/M$. Taking the ratio of these expressions gives the mass ratio of the system,

$$\frac{a_A}{a_B} = \frac{m_B}{m_A}$$

(see equation 7.1 of Ostlie & Carroll), and so we have $m_B/m_A = 0.466$. Combining this result with the calculated total mass, the masses of the individual components are found as $m_A = 4.39 \times 10^{30}\,\mathrm{kg} = 2.21\,M_\odot$ and $m_B = 2.04 \times 10^{30}\,\mathrm{kg} = 1.03\,M_\odot$.

(b). From equation 3.8 of Ostlie & Carroll,

$$M = M_\odot - 2.5 \log\left(\frac{L}{L_\odot}\right).$$

Solving for the luminosity,

$$\frac{L}{L_\odot} = 10^{(M_\odot - M)/2.5}$$

Plugging in the numbers, with $M_\odot = 4.74$, gives $L_a = 22.5\,L_\odot$ and $L_b = 0.0240\,L_\odot$.

(c). Assuming that Sirius B radiates as a blackbody, then

$$L_B = 4\pi R_B^2 \sigma T_B^4$$

Solving for the radius gives $R_B = 5.85 \times 10^6\,\mathrm{m} = 8.74 \times 10^{-4}\,R_\odot = 0.917\,R_\oplus$.

Q7.5 (a). For a circular-orbit binary system, equation 7.5 of Ostlie and Carroll gives

$$\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}},$$

while equation 7.6 gives

$$(m_1 + m_2)\sin^3 i = \frac{P}{2\pi G}(v_{1r} + v_{2r})^3.$$

Plugging in the supplied values of the period and maximum measured radial velocities gives $m_1 \sin^3 i = 1.15 \times 10^{31}\,\mathrm{kg} = 5.80\,M_\odot$ and $m_2 \sin^3 i = 5.67 \times 10^{30}\,\mathrm{kg} = 2.85\,M_\odot$.

(b). For randomly-oriented orbits, and taking into account the Doppler-shift selection effect, $\langle \sin^3 i \rangle \approx 2/3$. This gives mass estimates as $m_1 \approx 1.7 \times 10^{31}\,\mathrm{kg} \approx 8.7\,M_\odot$ and $m_2 \approx 8.5 \times 10^{30}\,\mathrm{kg} \approx 4.3\,M_\odot$.

Q7.6 (a). For a circular-orbit binary system, equation 7.5 of Ostlie and Carroll gives

$$\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}}.$$

Plugging in the numbers, we find $m_1/m_2 = 4.15$.

(b). Kepler's third law for circular orbits gives the sum of masses as

$$(m_1 + m_2)\sin^3 i = \frac{P}{2\pi G}(v_{1r} + v_{2r})^3$$

(cf. equation 7.6 of Ostlie & Carroll). Assuming $i = 90°$, plugging in the numbers gives $m_1 + m_2 = 1.02 \times 10^{31}\,\mathrm{kg} = 5.13\,M_\odot$.

(c). Combining the above numbers, we find that $m_1 = 8.23 \times 10^{30}\,\mathrm{kg} = 4.14\,M_\odot$ and $m_2 = 1.98 \times 10^{30}\,\mathrm{kg} = 1.00\,M_\odot$.

(d). The individual radii can be found from the timing of the eclipses. The time between first contact and minimum light gives the radius of the secondary (assuming that it's the smaller star), via

$$R_2 = \frac{v}{2}(t_b - t_a)$$

(cf. equation 7.8 of Ostlie & Carrol). For $i = 90°$, the relative velocity of the two stars during eclipse is $v = v_{1r} + v_{2r} = 27.8\,\mathrm{km s^{-1}}$, and so the secondary radius is found as $R_2 = 6.97 \times 10^8\,\mathrm{m} = 1.00\,R_\odot$.

The radius of the primary is found from the duration of the primary minimum, as

$$R_1 = \frac{v}{2}(t_c - t_b) + R_2.$$

Plugging in the numbers gives $R_1 = 1.47 \times 10^9\,\mathrm{m} = 2.11\,\mathrm{R}_\odot$.

(e). The ratio of effective temperatures can be found from the ratio of the eclipse depths:

$$\frac{T_2}{T_1} = \left(\frac{F_0 - F_1}{F_0 - F_2}\right)^{1/4} = \left(\frac{1 - F_1/F_0}{1 - F_2/F_0}\right)^{1/4}$$

(this comes from equation 7.11 of Ostlie & Carrol; but I've used F instead of B to denote the flux received here on earth. These are not surface fluxes, they are measured fluxes!). To find the eclipse depths, we use the observed bolometric magnitudes:

$$\frac{F_1}{F_0} = 10^{(m_{\mathrm{bol},0} - m_{\mathrm{bol},1})/2.5} = 0.0302,$$

$$\frac{F_2}{F_0} = 10^{(m_{\mathrm{bol},0} - m_{\mathrm{bol},2})/2.5} = 0.964.$$

Solving for the effective temperature ratio, we find $T_2/T_1 = 2.28$.

Bonus The energy expended can be calculated from the difference between the initial and final energy of SpaceShip One. In its initial state, at rest on the ground, it has a total (gravitational plus kinetic) eneregy per unit mass given by

$$E_{\mathrm{init}} = -\frac{GM_\oplus}{R_\oplus} + \frac{1}{2}v_{\mathrm{rot}}^2.$$

Here, v_{rot} is the velocity due to the Earth's rotation:

$$v_{\mathrm{rot}} = \frac{2\pi}{1\,\mathrm{dy}} R_\oplus \cos l,$$

where l is the launch latitude ($\cos l = \sin\theta$).

In the final state, with altitude A above the Earth's surface, the total energy per unit mass is given by

$$E_{\mathrm{final}} = -\frac{GM_\oplus}{R_\oplus + A} + \frac{1}{2}v_{\mathrm{tan}}^2,$$

where $v_{\rm tan}$ is the tangential velocity relative to the Earth's center. Note that there is no contribution from the radial velocity, since it goes to zero when the maximum altitude is reached. To find $v_{\rm tan}$, we make use of conservation of angular momentum, which requires that

$$v_{\rm rot} R_\oplus = v_{\rm tan}(R_\oplus + A)$$

Hence,

$$E_{\rm final} = -\frac{GM_\oplus}{R_\oplus + A} + \frac{1}{2}v_{\rm rot}^2 \left(\frac{R_\oplus}{R_\oplus + A}\right)^2$$

Taking the difference between the two expressions, the energy expended per unit mass is

$$\Delta E = E_{\rm final} - E_{\rm init} = -GM_\oplus \left(\frac{1}{R_\oplus + A} + \frac{1}{R_\oplus}\right) + \frac{1}{2}v_{\rm rot}^2 \left[\left(\frac{R_\oplus}{R_\oplus + A}\right)^2 - 1\right].$$

Plugging in the numbers, we find that $\Delta E = 1.08 \times 10^6$ Jkg^{-1}.

To compare against the equivalent value for launch into a circular orbit, note that for the circular-orbit case angular momentum will *not* be conserved (because we need to thrust in the tangential direction to reach orbit). However, for a circular orbit, the tangential velocity can be found by balancing centripetal and gravitational forces:

$$\frac{v_{\rm tan}^2}{R_\oplus + A} = \frac{GM_\oplus}{(R_\oplus + A)^2},$$

from which we have

$$\frac{1}{2}v_{\rm tan}^2 = \frac{GM_\oplus}{2(R_\oplus + A)}$$

The energy per unit mass required to reach orbit then becomes

$$\Delta E_{\rm orb} = -GM_\oplus \left(\frac{1}{R_\oplus + A} + \frac{1}{R_\oplus}\right) + \frac{GM_\oplus}{2(R_\oplus + A)} - \frac{1}{2}v_{\rm rot}^2.$$

Plugging in the numbers, we find that $\Delta E_{\rm orb} = 31.7 \times 10^6$ Jkg^{-1}. This is ~ 30 times more than the SpaceShip One flight, indicating that private spaceflight will likely be quite a bit more expensive than people might expect from examining the costs of the SpaceShip One project.