

Assignment 1 — Solutions [*Revision* : 1.3]

Q3.2 At a distance d from a light source with luminosity L , the radiant flux is given by

$$F = \frac{L}{4\pi d^2}.$$

Rearranging, we have

$$d = \sqrt{\frac{L}{4\pi F}}.$$

Then, with $L = 100 \text{ W}$ (the bulb's luminosity), and $F = 1365 \text{ Wm}^{-2}$ (the solar irradiance), we find that $d = 0.076 \text{ m} = 7.6 \text{ cm}$.

Q3.3 (a). (i) $d \text{ pc} = 1/p'' = 1/0.379'' = 2.64 \text{ pc}$

(ii) $2.64 \text{ pc} = 2.64 \times 3.26 \text{ ly} = 8.60 \text{ ly}$

(iii) $2.64 \text{ pc} = 2.64 \times 206266 \text{ AU} = 5.45 \times 10^5 \text{ AU}$

(iv) $5.45 \times 10^4 \text{ AU} = (5.45 \times 10^4) \times (1.5 \times 10^{11}) \text{ m} = 8.17 \times 10^{15} \text{ m}$

(b). Distance modulus $m - M = 5 \log(d/10 \text{ pc})$. With $d = 2.64 \text{ pc}$, $m - M = -2.89$

Q3.4 From the preceding question,

$$m_{\text{bol}} - M_{\text{bol}} = 5 \log(d/10 \text{ pc}) = -2.89.$$

From Example 3.6.1 of Ostlie & Carroll, $m_{\text{bol}} = -1.53$, from which we find $M_{\text{bol}} = 1.36$. Then using

$$M - M_{\odot} = -2.5 \log\left(\frac{L}{L_{\odot}}\right),$$

we find $L/L_{\odot} = 10^{(1.36-4.74)/-2.5} = 22.5$.

Q3.6 For any two stars,

$$m_1 - m_2 = -2.5 \log\left(\frac{F_1}{F_2}\right).$$

Let star 2 be the Sun placed at 10 pc; then, $m_2 = M_{\odot}$, while $F_2 = F_{10,\odot}$. Then,

$$m_1 - M_{\odot} = -2.5 \log\left(\frac{F_1}{F_{10,\odot}}\right).$$

Rearranging, and dropping the 1 subscripts,

$$m = M_{\odot} - 2.5 \log\left(\frac{F}{F_{10,\odot}}\right).$$

Q3.9 (a). The Stefan-Boltzmann equation gives the luminosity as

$$L = 4\pi R^2 \sigma T^4;$$

plugging in the supplied numbers, we obtain $L = 1.17 \times 10^{31} \text{ W} = 3.05 \times 10^4 L_{\odot}$.

(b). Applying equation (3.8) from Ostlie & Carroll,

$$M = M_{\odot} - 2.5 \log\left(\frac{L}{L_{\odot}}\right),$$

with the Sun's absolute bolometric magnitude $M_{\odot} = 4.74$, gives $M = -6.47$.

- (c). The absolute and apparent magnitudes are related via the distance modulus,

$$m - M = 5 \log(d/10 \text{ pc}).$$

With $d = 123 \text{ pc}$, we find $m = -1.01$.

- (d). Using the values found above, $m - M = 5.45$.

- (e). At the star's surface,

$$F_{\text{surf}} = \frac{L}{4\pi R^2} = \sigma T^4.$$

Plugging in the numbers, $F_{\text{surf}} = 3.49 \times 10^{10} \text{ W m}^{-2}$.

- (f). The radiant flux at the Earth's surface is given by

$$F = \frac{L}{4\pi d^2}$$

Plugging in the numbers, $F = 6.44 \times 10^{-8} \text{ W m}^{-2}$. This is a factor of $\sim 5 \times 10^{-11}$ smaller than the solar irradiance $F_{\odot} = 1365 \text{ W m}^{-2}$.

- (g). From Wien's law,

$$\lambda_{\text{max}} = \frac{0.0029 \text{ K m}}{T}.$$

Plugging in the numbers, $\lambda_{\text{max}} = 103 \text{ nm}$.

Q3.10 (a). The full Planck function is

$$B_{\lambda}(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}.$$

In the limit where $\lambda \gg hc/kT$ the argument of the exponential in the denominator is very small, and we can use the Taylor-series expansion

$$e^{hc/\lambda kT} \approx 1 + hc/\lambda kT.$$

Then, the Planck function becomes

$$B_{\lambda}(T) \approx \frac{2hc^2/\lambda^5}{1 + hc/\lambda kT - 1} \approx \frac{2ckT}{\lambda^4}.$$

This is the Rayleigh-Jeans law; it does not depend on Planck's constant h , and blows up in the short-wavelength limit.

- (b). See Fig. 1. The Rayleigh-Jeans value is twice as large as the Planck function at $\lambda \approx 2000 \text{ nm}$.

Q3.11 The full Planck function is

$$B_{\lambda}(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}.$$

In the limit where $\lambda \ll hc/kT$ the exponential in the denominator is very large, and the “-1” term can be neglected. Then, we have

$$B_{\lambda}(T) \approx 2hc^2\lambda^{-5}e^{-hc/\lambda kT},$$

which matches Wien's empirically derived relation

$$B_{\lambda}(T) \approx a\lambda^{-5}e^{-b/\lambda T}$$

(for appropriate choices of a and b).

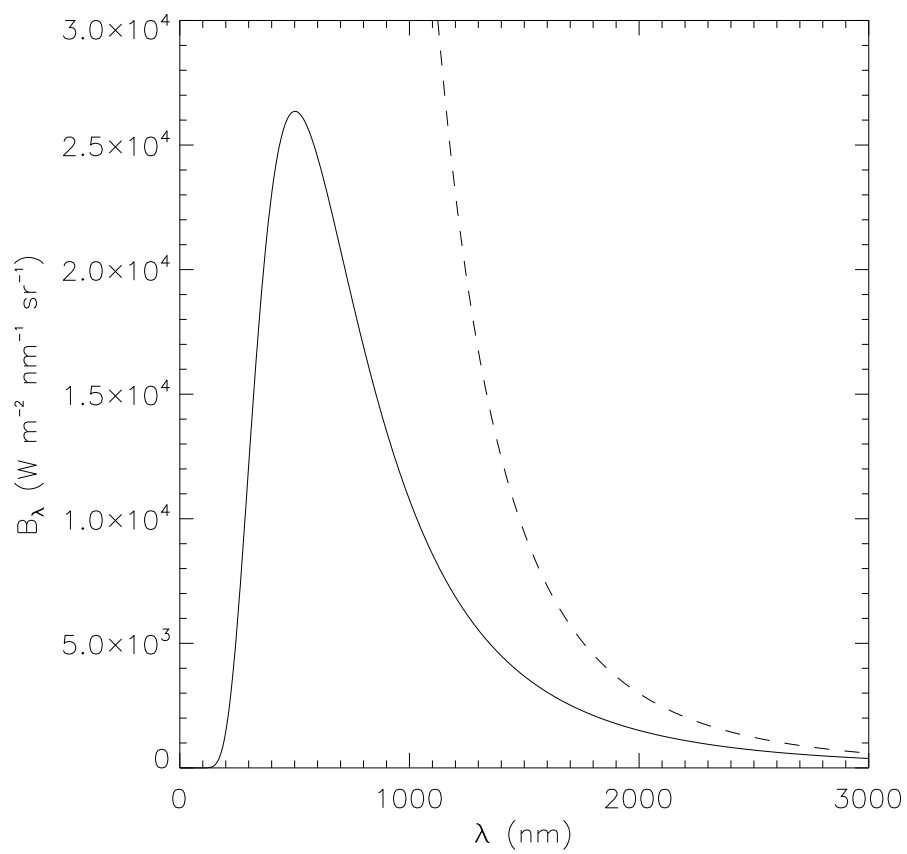


Figure 1: The Planck function for the Sun ($T_\odot = 5777 \text{ K}$), and the Rayleigh-Jeans approximation (dotted).

Q3.13 (a). The Planck function per unit frequency interval is

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

To find the frequency maximum ν_{\max} , we solve the equation

$$\frac{\partial B_\nu}{\partial \nu} = \frac{2h}{c^2} \left(\frac{3\nu^2}{e^{h\nu/kT} - 1} - \frac{h\nu^3/kT e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2} \right) = 0$$

Simplifying,

$$3(e^{h\nu/kT} - 1) - h\nu/kT e^{h\nu/kT} = 0$$

Introducing $u \equiv h\nu/kT - 3$, this becomes

$$ue^{u+3} + 3 = 0,$$

which can be rearranged as

$$ue^u = -3e^{-3}.$$

This is a nasty transcendental equation, whose solution involves the Lambert W function (see Wikipedia). A numerical approximation to the solution is

$$u = -0.179.$$

Substituting the definition of u back in, we have

$$h\nu/kT - 3 = -0.179,$$

which can be rearranged to get the final result

$$\nu_{\max} = 2.82 \frac{kT}{h} = 5.88 \times 10^{10} \text{ Hz K}^{-1} \cdot T.$$

(b). Applying the above expression to the Sun gives $\nu_{\max} = 3.40 \times 10^{14} \text{ Hz}$.

(c). The wavelength is $\lambda = c/\nu_{\max} = 880 \text{ nm}$; this is in the infra-red.

Q3.16 The filtered flux in some passband x is given by

$$F_x = \int_0^\infty F_\lambda S_x(\lambda) d\lambda.$$

If λ_x denotes the central wavelength of the passband, and $\Delta\lambda_x$ the full width, then this may be approximated by

$$F_x \approx F_{\lambda_x} \Delta\lambda_x$$

(see p. 78 of Ostlie & Carroll). Assuming that $F_\lambda \propto B_\lambda(T)$, with a blackbody temperature of 9,600 K, gives $F_U \propto 2.1$, $F_B \propto 2.4$, and $F_V \propto 1.5$ (with the same constant of proportionality). Even though the Planck function is largest in the U band, Vega appears brightest in the B band due to the larger bandwidth.

Q3.19 (a). Assuming a blackbody spectrum, the $U - B$ color is given by

$$U - B = -2.5 \log \left(\frac{B_{365} \Delta\lambda_U}{B_{440} \Delta\lambda_B} \right) + C_{U-B}$$

(see p. 78 of Ostlie & Carroll). Evaluating the Planck function $B_\lambda(T)$ for the surface temperature $T = 22,000 \text{ K}$ of Shaula gives $U - B = -1.1$. A similar procedure gives $B - V = -0.23$. The observed value of $U - B = -0.90$ is rather redder than the calculated value, due to departures of the star's spectrum from a true blackbody (see Fig. 3.11 of Ostlie & Carroll)

(b). The distance to Shaula is $d = 1/p'' = 216 \text{ pc}$; using the formula

$$V - M_V = 5 \log(d/10 \text{ pc})$$

then gives the absolute visual magnitude as $M_V = -5.05$.

Bonus The amount of solar radiation intercepted by the Earth every second is given by the product of the solar irradiance and the Earth's cross sectional area:

$$\frac{dE}{dt} = F_{\odot} \cdot \pi R_{\oplus}^2.$$

where F_{\odot} is, as usual, the solar constant. In thermal equilibrium, this must equal the amount re-radiated every second — that is, the Earth's luminosity. Assuming blackbody emission, this is given by

$$L_{\oplus} = 4\pi R_{\oplus}^2 \sigma T^4.$$

Setting $dE/dt = L_{\oplus}$ and solving for T gives

$$T = \left(\frac{F_{\odot}}{4\sigma} \right)^{1/4};$$

plugging in the numbers gives 278 K. This is rather smaller than the global average temperatures of $\sim 287 \text{ K}$ (14°C); the discrepancy is caused by the greenhouse effect (the discrepancy would be much larger if we had included the fact that much of the solar irradiance is reflected of clouds in the Earth's atmosphere).