34 — Degeneracy [*Revision* : 1.1]

- Breakdown of ideal gas law
  - So far, assumed gas pressure follows ideal equation of state:

$$P_{\rm gas} = \frac{\rho k T}{\mu m_{\rm H}}$$

  - However, at sufficiently high densities, and sufficiently low temperatures, ideal assumption begins to break down:
    * Number density of particles n is large
    * Typical momentum of particles is small
    * Particles concentrated in small volume of phase (momentum/position) space
    * But there's a limit to how tightly particles can be packed in phase space: **Pauli exclusion principle**
    * In limit where exlusion principle is important, gas is **degenerate**; different equation of state
- Non-relativistic degenerate gas
  - Consider Maxwell-Boltzmann velocity distribution function for (non-degenerate) particles of mass m at temperature T:

$$n_v {\rm d}v = n \left(\frac{m}{2\pi k T}\right)^{3/2} {\rm e}^{-mv^2/2kT} 4\pi v^2 {\rm d}v$$

    (number of particles per unit volume with velocities in interval $[v, v + {\rm d}v]$)
  - Can also be written as momentum distribution function:

$$n_p {\rm d}p = n \left(\frac{1}{2\pi m k T}\right)^{3/2} {\rm e}^{-p^2/2mkT} 4\pi p^2 {\rm d}p$$

  - In limit $T \to 0$, particles concentrated around $p = 0$; all try to have lowest possible momentum & energy $E(p)$, which is zero
  - However, this can run foul of Pauli exclusion principle, since all particles in same momentum state
  - In fact, lowest momentum/energy state has all particles filling up available cells of phase space, up to some maximum momentum $p_{\rm F}$ (the **Fermi momentum**)

$$n_p {\rm d}p = \begin{cases} \frac{4\pi p^2 {\rm d}p}{h^3/2} & p < p_{\rm F} \\ 0 & p > p_{\rm F} \end{cases}$$

    Numerator: volume of shell in phase space with momentum in interval $[p, p+{\rm d}p]$; denominator: volume in phase space occupied by each distinct quantum state (spin gives factor of 2)
  - Fermi momentum set by requirement

$$\int_0^{p_{\rm F}} \frac{4\pi p^2 {\rm d}p}{h^3/2}\,{\rm d}p = n,$$

    so that

$$p_{\rm F} \sim n^{1/3}$$

    (henceforth, drop all unimportant factors in expressions, for simplicity)

- To obtain equation of state, recall that pressure scales as

$$P_{\rm gas} \sim u$$

  where u is kinetic energy density

- To find u, integrate over all occupied states

$$u \sim \int_0^{p_{\rm F}} E(p) n_p \, {\rm d}p$$

  If gas is non-relativistic, $E(p) = p^2/2m$; hence,

$$u \sim \int_0^{p_{\rm F}} p^2 p^2 \, {\rm d}p \sim p_{\rm F}^5$$

  and

$$P_{\rm gas} \sim p_{\rm F}^5 \sim n^{5/3}$$

- With $\rho \sim n$, final result:

$$P_{\rm gas} \sim \rho^{5/3}$$

- Discussion:
  * This is **degeneracy pressure**; comes from exlusion principle rather than thermal motions
  * Independent of temperature
  * Polytropic — a fully degenerate star is a polytrope
  * Strictly applies at zero temperature, but becomes good approximation when average particle momenta are below $p_{\rm F}$
  * This limit equivalent to inequality

$$kT \lesssim \frac{p_{\rm F}^2}{2m} \equiv E_{\rm F}$$

    ($E_{\rm F}$ is **Fermi energy**). Note mass dependence; lighter particles become denerate first
  * Often see this inequality written as

$$\frac{T}{rho^{2/3}} < \mathcal{D}$$

    where $\mathcal{D}$ depends on m
  * **Electron degeneracy** important in cores of low-mass stars, and in white dwarfs
  * **Neutron degeneracy** important in neutron stars (no electrons left!)
  * Degeneracy responsible for **helium flash**: when helium ignites, temperature increases (particles move faster), but pressure does not increase (because degeneracy condition above still holds); no 'safety valve' where star expands to cool off reactions, thus runaway burning

- Relativistic degenerate gas
  - Similar to derivation above, but must use Einstein energy-momentum relation

$$E^2 = p^2 c^2 + m_0^2 c^4$$

    (m_0 is rest mass)

- In relativistic limit $E \gg m_0 c^2$, $E \sim p$
- Hence,

$$u \sim \int_0^{p_\mathrm{F}} p p^2 \, \mathrm{d}p \sim p_\mathrm{F}^4$$

  and

$$P_\mathrm{gas} \sim \rho^{4/3}$$

- Electrons become relativistic in white dwarf stars as they approach **Chandrasekhar limit** (limiting mass, above which there is collapse to neutron star)