33 — The Hayashi Line [$Revision$: 1.2]

- Luminosity scaling laws
  - Back in Notes 23, derived luminosity scaling relation:

    $$L \sim \frac{ac}{\bar{\kappa}} \left(\frac{G\mu m_H}{k} \right)^4 M^3$$

    General result: $L \propto M^3$, but also M increases with μ (explaining luminosity increase on main sequence)
  - Big assumption: star is completely radiative
  - Not too bad for 'hot end' of Hertzsprung-Russell diagram
  - But once stars move across to $T_{\rm eff} \lesssim 5{,}000\,{\rm K}$, strong convection zone forms due to H^- opacity (electron loosely bound to hydrogen atom)
  - So, what is luminosity scaling relation for fully-convective star?
  - Recap: for radiative diffusion,

    $$L_{\rm rad} = \frac{16\pi r^2 acT^4}{3\bar{\kappa}\rho} H_P \nabla \propto \nabla$$

    where

    $$\nabla \equiv \frac{{\rm d}\ln T}{{\rm d}\ln P}$$

    is the temperature gradient in star. Luminosity determined by pressure-temperature structure (which in turn comes from hydrostatic equilibrium)
  - But for convective transport,

    $$L_{\rm conv} = 4\pi r^2 C_P T \rho (\nabla - \nabla_{\rm ad})^{3/2} \alpha^2 \left(2\beta g H_P\right)^{1/2} \propto (\nabla - \nabla_{\rm ad})^{3/2}$$

  - Because convection is v. efficient, constant of proportionality is v. large; only a tiny excess of ∇ above $\nabla_{\rm ad}$ is required to transport all luminosity (so $\nabla \approx \nabla_{\rm ad}$)
  - So, luminosity is almost completely decoupled from temperature-pressure structure. How do we then compute it?

- Fully-convective stars
  - Always will have radiative atmosphere at surface, because convection becomes inefficient there
  - Can use surface layers to constrain temperature-pressure structure inside
  - Start by finding pressure at photosphere ($\tau = 2/3$, $T = T_{\rm eff}$)
  - In atmosphere, optical depth given by

    $$\tau = \int_r^\infty \bar{\kappa}\rho\,{\rm d}r,$$

    so if $r = R$ at $\tau = 2/3$,

    $$\frac{2}{3} = \int_R^\infty \bar{\kappa}\rho\,{\rm d}r$$

- Assume opacity constant in atmosphere:

$$\frac{2}{3} = \bar{\kappa} \int_R^\infty \rho \,\mathrm{d}r$$

  but from hydrostatic equilibrium:

$$P(r=R) = \int_R^\infty g\rho \,\mathrm{d}r \approx \frac{GM}{R^2} \int_R^\infty \rho \,\mathrm{d}r$$

- So,

$$P(r=R) \approx \frac{GM}{R^2} \frac{2}{3} \frac{1}{\bar{\kappa}}$$

- From star to star, $\bar{\kappa}$ will itself vary with temperature and pressure; assume

$$\bar{\kappa} = \bar{\kappa}_0 P^a T^b,$$

  and so

$$P(r=R) \approx \frac{GM}{R^2} \frac{2}{3} \frac{P^{-a} T_{\mathrm{eff}}^{-a}}{\bar{\kappa}_0}$$

  which solves as

$$P(r=R) \approx \left(\frac{GM}{R^2} \frac{2}{3\bar{\kappa}_0} T_{\mathrm{eff}}^{-b} \right)^{\frac{1}{1+a}}$$

- So, we now know pressure and temperature at base of atmosphere. This must match smoothly onto interior solution
- Assume in interior $\nabla = \nabla_{\mathrm{ad}}$, so

$$P = C\,T^{1/\nabla_{\mathrm{ad}}};$$

  but an ideal-gas polytrope has

$$P = K^{-n} \left(\frac{k}{\mu m_H} \right)^{1+n} T^{1+n}$$

  and so we have a polytrope with $n = 1/nablaad - 1 = 3/2$
- Recall for polytropes:

$$r = \lambda_n \xi,$$

  where

$$\lambda_n = \left[(n+1) \left(\frac{K \rho_{\mathrm{c}}^{1/n-1}}{4\pi G} \right) \right]^{1/2}$$

  Also,

$$M = -4\pi \lambda_n^3 \rho_{\mathrm{c}} \xi_1^2 \left. \frac{\mathrm{d}D_n}{\mathrm{d}\xi} \right|_{\xi=\xi_1}.$$

- Set $r = R$ in these expressions, solve for K scaling

$$K \sim M^{1/3} R$$

- Hence, in interior

$$P = C' R^{-3/2} M^{-1/3} T^{5/2}$$

  where C' depends only on μ and n

- To match surface solution, require that

$$\left(\frac{GM}{R^2}\frac{2}{3\bar{\kappa}_0}T_{\rm eff}^{-b}\right)^{\frac{1}{1+a}} = C' R^{-3/2} M^{-1/3} T_{\rm eff}^{5/2}$$

  For given M, this describes a one-to-one relationship between $T_{\rm eff}$ and R — i.e., a curve in HRD

- After some math, with

$$L = 4\pi R^2 \sigma T_{\rm eff}^4,$$

  can be rearranged as

$$\ln T_{\rm eff} = A \ln L + B \ln M + \text{const.}$$

  where

$$A = \frac{0.75a - 0.25}{b + 5.5a + 1.5}, \qquad B = \frac{0.5a + 1.5}{b + 5.5a + 1.5}$$

- The Hayashi Line
  - Need values for a and b; for H^- opacity, good approximation is $a \approx 1$, $b \approx 3$, and so

$$A = 0.05, \qquad B = 0.2$$

  - Nearly vertical line in HRD
  - Weak dependence on mass means that all fully-convective stars lie on same **Hayashi line**
  - Locus of stars that are fully convective
  - No stars allowed to right of line; **forbidden region**