32 — Post Main-Sequence Evolution [*Revision* : 1.1]

- Shell Ignition
  - Shortly before hydrogen exhaustion in center of stars above $\sim 1.1\,M_\odot$, energy produced by core is insufficient to match luminosity radiated by envelope
  - Typically occurs when $X \approx 0.05$
  - Kelvin-Helmholtz contraction makes up luminosity difference; star moves to higher $T_{\rm eff}$ on HR diagram — little 'jag' in tracks
  - For stars below $\sim 1.1\,M_\odot$, no need for KH contraction; no jag
  - In both cases, main sequence evolution ends when hydrogen exhausted at center ($X \to 0$)
  - Then, a new energy source appears
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    * Core is made of 'inert' helium; relative mass $M_{\rm core}/M$ increases with M
    * Core is hot because of high μ
    * Core wants to be isothermal because $\epsilon = 0$ and so $F_{\rm rad} = 0$ and so $\mathrm{d}T/\mathrm{d}r = 0$
    * Temperatures at core boundary high enough to burn hydrogen
    * Shell ignition occurs!
    * Shell appearance smooth for low-mass stars ($M \lesssim 1.3\,M_\odot$)
    * Shell appearance abrupt for higher-mass stars ($M \gtrsim 1.3\,M_\odot$)
  - As shell burning continues, it adds helium to core; core mass grows
- Evolution to the Red
  - The envelope of the star had previously adjusted self for core burning
  - But now we have new, different energy source — shell burning
  - Shell burning typically produces more luminosity than core burning (higher temperature)
  - Shell luminosity is greater than envelope can typically radiate
  - So, envelope absorbs excess luminosity, heats up and expands
  - Effective temperature decreases; star moves across to base of **red giant branch** (RGB) in Hertzsrpung-Russell diagram
  - Typically, timescale for evolution across HR diagram $\sim$ shell burning timescale...
  - ...if it weren't for a problem in the core
  - Sidenote: astronomers still argue about why stars form red giants; all models show it, but *why* they show it is subject of great debate
- The Schönberg-Chandrasekhar limit
  - So far, we've neglected what happens in core
  - Initially, core is isothermal (although exceptions occur for high-mass stars $M \gtrsim 6\,M_\odot$)
  - Core does not need to KH contract because not losing energy through boundary; kept warm by shell burning
  - However, core is growning in mass
  - Can it always support pressure of overlying envelope?
  - Assume temperature $T_{\rm core}$ within core (and at core boundary) is constant
  - Assume downward pressure of envelope at core boundary, $P_{\rm env}$ is constant (can show it depends mainly on mass anbd $T_{\rm core}$)

- Use virial theorem derivation to calculate upward pressure of core $P_{\rm core}$:
  * Consider core in hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho = -\frac{GM_r}{r^2}\rho$$

  * Multiply both sides by $4\pi r^3$, integrate over *core*:

$$\int_0^{R_{\rm core}} \frac{\mathrm{d}P}{\mathrm{d}r} 4\pi r^3 \,\mathrm{d}r = -\int_0^{R_{\rm core}} \frac{GM_r}{r^2} 4\pi r^3 \rho \,\mathrm{d}r = E_{\rm g,core}$$

    where $E_{\rm g,core}$ is gravitational energy of core
  * Do integral on lhs by parts:

$$\left[4\pi r^3 P\right]_0^{R_{\rm core}} - 3\int_0^{R_{\rm core}} P 4\pi r^2 \,\mathrm{d}r = E_{\rm g,core}$$

  * Use ideal gas law:

$$4\pi R_{\rm core}^3 P_{\rm core} - \frac{3M_{\rm core}kT}{\mu m_H} = E_{\rm g,core}$$

  * To calculate $E_{\rm g,core}$, assume density is equal to mean density in core:

$$\rho \approx \frac{3M_{\rm core}}{4\pi R_{\rm core}^3}$$

    Then,

$$E_{\rm g,core} \approx -\frac{3}{5}\frac{GM_{\rm core}^2}{R_{\rm core}}$$

  * Solving for $P_{\rm core}$:

$$P_{\rm core} = \frac{3}{4\pi R_{\rm core}^3}\left(\frac{M_{\rm core}kT_{\rm core}}{\mu m_H} - \frac{1}{5}\frac{GM_{\rm core}^2}{R_{\rm core}}\right)$$

  * This core pressure $P_{\rm core}$ must match the envelope pressure $P_{\rm env}$; the core adjusts its radius to make this so
  * But can a match always be found? Not necessarily, because $P_{\rm core}$ has a local maximum, above which it cannot go
  * To find maximum, differentiate $P_{\rm core}$ wrt to $R_{\rm core}$, & set to zero; find

$$R_{\rm core,max} = \frac{4}{15}\frac{GM_{\rm core}\mu m_H}{kT_{\rm core}}$$

    and

$$P_{\rm core,max} = \frac{10125}{1024 G^3 M_{\rm core}^2}\left(\frac{kT_{\rm core}}{\mu m_H}\right)^4$$

    (Note: the derivation in O&C is wrong. Their answer is dimensionally correct, but their reasoning doesn't make sense)
  * Interesting result: as the core mass increases, eventually $P_{\rm core,max} < P_{\rm env}$, and there is no way the core can support envelope
  * This limiting core mass is called the **Schönberg-Chandrasekhar limit**

* Using result

  $$P_{\rm env} \propto \frac{T_{\rm core}^4}{M^2}$$

  (see Kippenhahn & Weigert) can show that SC limit reached at fractional mass

  $$\frac{M_{\rm core}}{M} \approx 0.37 \left(\frac{\mu_{\rm env}}{\mu_{\rm core}}\right)^2$$

  For $\mu_{\rm env} \approx 0.6$ (fully-ionized solar), and $\mu_{\rm core} \approx 1.3$ (all hydrogen turned to helium), find SC limit as

  $$\frac{M_{\rm core}}{M} \approx 0.08$$

- Above SC limit, core begins to contract rapidly
- Contraction heats core, sets up temperature gradients, makes core radiate
- Whole process happens on KH timescale — very quick!
- Means that evolution to red is much quicker than nuclear; leads to observational gap in distribution of stars in HRD (the **Hertzsprung gap**
- **Important**: SC limit is not cause of evolution to base of RGB; but it selts timescale
- Take note of special cases:
  * For low-mass stars ($M \lesssim 1.3\,M_\odot$), helium core is partly degenerate; higher pressure than predicted by above (ideal gas) argument, SC limit does not apply and evolution to RGB is slower
  * For higher-mass stars ($M \gtrsim 2-3\,M_\odot$), core is already above SC limit at end of main sequence; so, contraction happens immediately, isothermal core can never form