29 — Stellar Structure & Evolution $[Revision : 1.1]$

- Equations of Stellar Structure
  - In previous notes, we have derived governing equations of stellar structure
  - These equations must be solved together to find structure of star
  - Reminder:
    * Hydrostatic equilibrium:
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho$$
    * Poisson's equation (solution of):
$$g = \frac{GM_r}{r^2}$$
    * Mass-radius:
$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho$$
    * Energy conservation:
$$\frac{\mathrm{d}L_r}{\mathrm{d}r} = 4\pi r^2 \rho\epsilon$$
    * Energy transport:
$$L_r = 4\pi r^2 (F_{\mathrm{rad}} + F_{\mathrm{conv}})$$
    * Transport by radiative diffusion:
$$F_{\mathrm{rad}} = -\frac{4acT^3}{3\bar{\kappa}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$
    * Transport by convection:
$$F_{\mathrm{conv}} = C_P T\rho(\nabla - \nabla_{\mathrm{ad}})^{3/2}\alpha^2\,(2\beta g H_P)^{1/2}$$
  - Also need constitutive relations:
    * Equation of state:
$$P = \frac{\rho k T}{\mu m_H} + \frac{aT^4}{3}$$
      with $\mu = \mu(\rho, T, \text{composition})$ found from Saha equation. Equation becomes more complicated at high ρ due to degeneracy
    * Rosseland mean opacity:
$$\bar{\kappa} = \bar{\kappa}(\rho, T, \text{composition})$$
      Contributions to $\bar{\kappa}$ from bound-bound, bound-free, free-free and electron scattering processes. Look up from pre-calculated tables
    * Energy generation rate:
$$\epsilon = \epsilon(\rho, T, \text{composition})$$
      Contributions to ϵ from nuclear burning & (negative) neutrino losses (look up from pre-calculated tables). Also, contribution from Kelvin-Helmholtz contraction:
$$\epsilon_{\mathrm{KH}} = -T\frac{\mathrm{d}S}{\mathrm{d}t}$$
      where S is entropy per unit mass (specific entropy). Note time derivative — this term only kicks in when star is contracting & heating up

- To solve these equations, need to apply boundary conditions
  * At center $r \longrightarrow 0$:
$$M_r \longrightarrow 0$$
$$L_r \longrightarrow 0$$
    by definition
  * At surface $r \longrightarrow R$:
$$M_r \longrightarrow M$$
$$\rho \longrightarrow 0$$
$$T \longrightarrow 0$$
    (and P goes to zero from equation of state). More realistic: match interior solution to model atmosphere

- Vogt-Russell Theorem & Evolution
  - Only two free parameters in solving equation: total mass M and initial chemical composition (assumed to be uniform). To satisfy all equations and boundary conditions simultaneously requires specific choice of luminosity L and radius R — these are **eigenvalues** of the problem
  - This concept expressed in **Vogt-Russel theorem**: *The mass and the composition structure throughout a star uniquely determine its radius, luminosity, and internal structure, as well as its subsequent evolution.*
  - As a corollary, *The dependence of a star's evolution on mass and composition is a consequence of the change in composition due to nuclear burning.* So, as star burns, the internal composition changes, and so also must internal structure.
  - Note: here we neglect rotation, magnetic fields, binarity, etc

- Solving the Equations
  - Solve via numerical integration of equations
  - Overall principle: replace derivatives by finite differences on grid:
$$\frac{\mathrm{d}P}{\mathrm{d}r} \longrightarrow \frac{P_{i+1} - P_i}{r_{i+1} - r_i}$$
    where $i = 1, 2, 3, \ldots$ indexes radial grid points
  - Simplest method: **shooting**
  - Guess values for central temperature, density, pressure
  - Integrate away from center: e.g.,
$$P_{i+1} = P_i + \frac{\mathrm{d}P}{\mathrm{d}r}(r_{i+1} - r_i),$$
    where $\mathrm{d}P/\mathrm{d}r$ is calculated from structure equations
  - Stop integration when surface is reached (e.g., $\rho \to 0$)
  - Solution won't satisfy all surface boundary conditions; so, adjust central values accordingly and repeat
  - Iterate until a self-consistent solution found

- Evolving the Solution

- Similar numerical procedure is used to evolve solution over time
- On main sequence, main time dependence of structure comes from hydrogen burning:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\epsilon_{\mathrm{nuclear}}}{26.7\,\mathrm{MeV}}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{\mathrm{d}X}{\mathrm{d}t}$$

  where $\epsilon_{\mathrm{nuclear}}$ is total energy generation rate (including neutrino production).
- Evolve abundances as

$$X^{n+1} = X^n + \frac{\mathrm{d}X}{\mathrm{d}t}(t^{n+1} - t^n)$$

  where $n = 1, 2, 3, \ldots$ indexes times
- Abundances can also evolve due to convective mixing; important in cores of massive stars