27 — Nuclear Energy Generation [$Revision: 1.2$]

- Energy Sources
  - Age of Moon from rocks: 4.5 billion years
  - Recall from Notes 19: Kelvin-Helmholtz timescale
    $$t_{\rm KH} = \frac{GM^2}{LR}$$
  - For Sun, $t_{\rm KH} \sim 30\,{\rm Myr}$, much shorter than age of Earth or solar system. So, the Sun cannot be powered by KH contraction
  - Likewise, Sun cannot be powered by chemical energy, because energy release per atom is on order of $\sim 1\,{\rm eV}$
  - Only possible energy source is **nuclear**; recognized in early 20$^{\rm th}$ Century, before details worked out
- Binding Energy
  - Energy is released when nucleus is assembled from constituent nucleons; this is **binding energy** of nucleus
  - Release comes from work done by **strong nuclear force**
  - Express binding energy as
    $$E_{\rm b} = \Delta m c^2$$
  - Δm is **mass defect** — difference in combined masses of constituent nucleons, m_j, and mass of nucleus $m_{\rm nuc}$:
    $$\Delta m = \sum_j m_j - m_{\rm nuc}$$
  - Classifying nuclei:
    * Z is number of protons (**atomic number**)
    * N is number of neutrons
    * A is number of nucleons (protons + neutrons; **mass number**)
    * $A = Z + N$
  - Write mass of constituents as
    $$\sum_j m_j = Z m_{\rm p} + N m_{\rm n} = Z m_{\rm p} + (A - Z) m_{\rm n},$$
    where $m_{\rm p}$ is proton mass, $m_{\rm n}$ is neutron mass
  - So, binding energy is
    $$E_{\rm b} = \left[Z m_{\rm p} + (A - Z) m_{\rm n} - m_{\rm nuc} \right] c^2$$
  - Binding energy per nucleon is
    $$\frac{E_{\rm b}}{A} = \frac{\left[Z m_{\rm p} + (A - Z) m_{\rm n} - m_{\rm nuc} \right] c^2}{A};$$
    indication of how tightly bound each nucleon is, and so how much energy can be extracted through nuclear reactions
  - If a reaction leads to net increase in $E_{\rm b}/A$, then it is **exothermic**
  - Graph of $E_{\rm b}/A$ against A is peaked at Fe ($\sim 8.5\,{\rm MeV}$

  - For A smaller than 56, use **fusion** to release energy
  - For A greater than 56, use **fission** to release energy

- Thermonuclear Reactions
  - Nuclear reactions occur by collisions between nuclei and/or protons ('free' neutrons decay into protons in $\sim$ 15 mins)
  - To get particles close enough to react via short-range strong force, must overcome repulsion between positive charges
  - Define close enough as radius of nucleus r_{nuc}
  - At this distance, Coulomb (electrostatic) potential energy of two particles is

$$U_{\text{C}} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r_{\text{nuc}}}$$

  - This energy must come from kinetic energy of particles, and in turn from thermal energy. So, for **thermonuclear reactions**

$$\frac{3}{2}kT \approx U_{\text{C}} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r_{\text{nuc}}}$$

  - To find r_{nuc}, use de Broglie wavelength

$$r_{\text{nuc}} \sim \lambda \sim \frac{h}{p}$$

    But

$$\frac{p^2}{2m} = \frac{3kT}{2}$$

    and so

$$r_{\text{nuc}} = \frac{h}{\sqrt{3mkT}}$$

  - Solving for T,

$$T = \frac{Z_1^2 Z_2^2 e^4 m}{12\pi^2 \epsilon_0^2 h^2 k}$$

    For proton-proton reaction, $Z_1 = Z_2 = 1$; but resulting T much higher than found e.g. in Sun
  - Solution to paradox: **quantum-mechanical tunneling**!

- Reaction rates
  - Consider reaction where target particle x interacts with incoming particles i
  - Number of reactions per target in time $\mathrm{d}t$:

$$N = \sigma v n_i \mathrm{d}t$$

    where σ is reaction cross section, v is velocity of incoming particles, and n_i is their number density
  - **HOWEVER**, not all incoming particles have same energy; and σ strongly dependent on energy

- So, consider number of reactions per second per target with incoming particle in energy interval $(E, E + \mathrm{d}E)$:

$$\mathrm{d}N_E = \sigma(E) v(E) n_{i,E} \, \mathrm{d}E \, \mathrm{d}t$$

  where $n_{i,E}$ is number of incoming particles with energy in same interval,

$$v(E) = \left(\frac{2E}{m_i} \right)^{1/2}$$

  where m_i is incoming particle mass
- Assume incoming particles are in thermal equilibrium; number distribution given by Maxwell-Boltzmann distribution:

$$n_{i,E} \mathrm{d}E = \frac{n_i}{n} n_E \mathrm{d}E$$

  where

$$n_E \, \mathrm{d}E = \frac{2n}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} E^{1/2} \mathrm{e}^{-E/kT} \, \mathrm{d}E$$

- Now must find energy-dependent cross-section $\sigma(E)$. Generally, complicated, but can at least get scalings
- For an actual interaction to take place, incoming particle must (a) 'hit' target particle, and (b) interact with it
- For (a), cross section of target varies with geometrical cross section:

$$\sigma(E) \propto \pi \lambda^2 \propto E^{-1}$$

  where λ is de Broglie wavelength
- For (b), cross section of target varies with tunneling probability. This probability is exponential with height of Coulomb barrier:

$$\sigma(E) \propto \mathrm{e}^{-2\pi^2 U_\mathrm{C}/E}$$

  (factor of $2\pi^2$ is from QM). Evaluate barrier height U_C from Coulomb potential at $r = \lambda$:

$$\frac{U_\mathrm{C}}{E} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 \lambda E}$$

  Final result:

$$\sigma(E) \propto \mathrm{e}^{-bE^{-1/2}}$$

  where

$$b = \frac{\pi m_i^{1/2} Z_1 Z_2 e^2}{2^{1/2} \epsilon_0 h}$$

- Putting (a) and (b) together:

$$\sigma(E) = \frac{S(E)}{E} \mathrm{e}^{-bE^{-1/2}}$$

  where function $S(E)$ is now some hopefully slowly-varying function of energy
- Reactions per second thus becomes

$$\mathrm{d}N_E = \frac{S(E)}{E} \mathrm{e}^{-bE^{-1/2}} \left(\frac{2E}{m_i} \right)^{1/2} \frac{2n}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} E^{1/2} \mathrm{e}^{-E/kT} \, \mathrm{d}E \, \mathrm{d}t$$

– Integrate over all energies, multiply by n_x divide by dt to get overall reaction rate per unit volume

$$r_{ix} = n_x \frac{\int \mathrm{d}N_E}{\mathrm{d}t} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(m_i \pi)^{1/2}} \int_0^\infty S(E) \mathrm{e}^{-bE^{-1/2}} \mathrm{e}^{-E/kT} \, \mathrm{d}E.$$

– Integrand is product of rapidly dropping factor $\mathrm{e}^{-E/kT}$ (due to decline in number of particles with high thermal energies), and rapidly growing factor $\mathrm{e}^{-bE^{-1/2}}$ (due to increased probability of barrier penetration at high energies). Two factors combine to form strongly peaked curve — the **Gamow peak**

– Top of Gamow peak has energy

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3}$$

Major contribution to rate integral over narrow energy interval centered on E_0; allows us to replace $S(E)$ with value at peak:

$$r_{ix} \approx n_x \frac{\int \mathrm{d}N_E}{\mathrm{d}t} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(m_i \pi)^{1/2}} S(E_0) \int_0^\infty \mathrm{e}^{-bE^{-1/2}} \mathrm{e}^{-E/kT} \, \mathrm{d}E.$$

However, this assumes $S(E)$ is slowly-varying function of energy; in fact, sometimes sharp variations occur due to **resonances**

– One missing ingredient in above analysis: **electron screening** makes nuclei appear 'less positive', and makes reactions easier to occur

– Above expressions for reaction rate are complicated; often fit using simple power law

$$r_{ix} \approx r_0 X_i X_x \rho^{\alpha'} T^{\beta}$$

If each reaction creates energy $\mathcal{E}_0$, energy generation rate per unit mass given by

$$\epsilon_{ix} = \frac{\mathcal{E}_\prime}{\rho} r_{ix} = \epsilon'_0 X_i X_x \rho^{\alpha} T^{\beta}$$