26 — Convective Transport $[Revision : 1.3]$

- Temperature Gradient
  - Recall from Notes 25:
    $$\nabla = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)$$
    (a measure of temperature gradient);
    $$\nabla_{\mathrm{ad}} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}} = \frac{\gamma-1}{\gamma}$$
    (a measure of thermodynamic properties of gas);
    $$\nabla_{\mathrm{rad}} = \frac{3\kappa\rho F}{4acT^4}\,H_P$$
    (a measure of total flux).
  - Condition for onset of convection: $\nabla_{\mathrm{rad}} > \nabla_{\mathrm{ad}}$
  - When convection sets in, how do we evaluate temperature gradient ∇?
  - Split total flux into radiative and convective parts:
    $$F = F_{\mathrm{rad}} + F_{\mathrm{conv}}$$
    or
    $$F_{\mathrm{rad}} = F - F_{\mathrm{conv}} = F\left(1 - \frac{F_{\mathrm{conv}}}{F}\right)$$
  - Multiply both sides:
    $$\frac{3\kappa\rho F_{\mathrm{rad}}}{4acT^4}\,H_P = \frac{3\kappa\rho F}{4acT^4}\,H_P\left(1 - \frac{F_{\mathrm{conv}}}{F}\right)$$
    or
    $$\nabla = \nabla_{\mathrm{rad}}\left(1 - \frac{F_{\mathrm{conv}}}{F}\right)$$
  - So, we can calculate ∇ from ∇_{rad} if we know what fraction of total flux is carried by convection
  - **Observe**: $\nabla \leq \nabla_{\mathrm{rad}}$ from above, but $\nabla \geq \nabla_{\mathrm{ad}}$ for convection to continue.
- Convective flux
  - To date, no comprehensive, self-consistent theory for calculating F_{conv}
  - However, some progress can be made with **mixing-length theory**
  - Return to bubble from Notes 25. Assume bubble rises distance
    $$\Delta r = \ell = \alpha H_P$$
    before it dissolves into its surroundings; ℓ is **mixing length**, usually representend in terms of dimensionless **mixing length parameter** α
  - Heat released to surroundings, per unit volume, is
    $$\delta q = C_P\rho\left(\Delta T^{(\mathrm{b})} - \Delta T^{(\mathrm{s})}\right)$$
    where C_P is specific heat at constant pressure, $\Delta T^{(\mathrm{b})}$ is change in temperature of blob (between start and end locations), and $\Delta T^{(\mathrm{b})}$ is change in temperature of surroundings

- For surroundings:

$$\frac{\Delta T^{(\mathrm{s})}}{T} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}\right)\frac{\Delta r}{r} = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r}\frac{\Delta r}{r} = -\nabla\frac{r}{H_P}\frac{\alpha H_P}{r} = \nabla\,\alpha$$

- For adibatic bubble motion:

$$\frac{\Delta T^{(\mathrm{b})}}{T} = \left(\frac{\partial\ln T}{\partial\ln P}\right)_{\mathrm{ad}}\frac{\mathrm{d}\ln P}{\mathrm{d}\ln r}\frac{\Delta r}{r} = \nabla_{\mathrm{ad}}\,\alpha$$

  (note generalized definition of ∇_{ad}!)

- So, heat released is

$$\delta q = C_P T\rho\,(\nabla - \nabla_{\mathrm{ad}})\,\alpha$$

- If bubble moves with average velocity v_{conv}, convective flux is

$$F_{\mathrm{conv}} = \delta q v_{\mathrm{conv}} = C_P T\rho\,(\nabla - \nabla_{\mathrm{ad}})\,\alpha v_{\mathrm{conv}}$$

  ...which is desired result

- Convective velocity
  - What sets the velocity?
  - Recall: net force per unit volume on bubble is

$$f = g\left(\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{1}{\gamma}\frac{\rho}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right)\Delta r$$

  - After a bit of algebra:

$$f = g\rho\,(\nabla - \nabla_{\mathrm{ad}})\,\frac{\Delta r}{H_P}$$

  - Take average in moving from $\Delta r = 0$ to $\Delta r = \ell$ is

$$\langle f\rangle = g\rho(\nabla - \nabla_{\mathrm{ad}})\frac{\alpha}{2}$$

  - Work done per unit volume on bubble:

$$w = \langle f\rangle\ell = g\rho(\nabla - \nabla_{\mathrm{ad}})\frac{\alpha^2}{H_P}$$

  - Assume some fraction β of this work goes into bubble kinetic energy; so

$$\frac{\rho v_{\mathrm{conv}}^2}{2} = \beta g\rho(\nabla - \nabla_{\mathrm{ad}})\alpha^2 H_P,$$

    or

$$v_{\mathrm{conv}} = \left[2\beta g(\nabla - \nabla_{\mathrm{ad}})\alpha^2 H_P\right]^{1/2}$$

- Convection equations
  - Combine v_{conv} expression with equation for convective flux:

$$F_{\mathrm{conv}} = C_P T\rho(\nabla - \nabla_{\mathrm{ad}})^{3/2}\alpha^2\,(2\beta g H_P)^{1/2}$$

    (Looks rather different to eqn. 10.100 of O&C, but should be essentially the same).

- Also have

$$\nabla = \nabla_{\mathrm{rad}} \left(1 - \frac{F_{\mathrm{conv}}}{F}\right)$$

- Two equations, two unknowns; solve for ∇ and F_{conv}

- Some Discussion
  - Suppose *all* flux is transported by convection; then, can calculate superadiabaticity:

$$\nabla - \nabla_{\mathrm{ad}} = \left[\frac{F}{C_P T \rho} \left(\frac{1}{2\beta g H_P}\right)^{1/2} \frac{1}{\alpha^2}\right]^{2/3}$$

  - Deep in star, temperature and density large; temperature gradient close to adibatic ($\nabla \approx \nabla_{\mathrm{ad}}$)
  - In outer layers, temperature gradient quite superadiabatic; convection is inefficient because material cannot hold much heat
  - Very close to surface, convective velocity becomes supersonic; signals breakdown of mixing length theory
  - Also, in reality bubble motion is not adiabatic; must account for radiative losses to surrounding medium, and this complicates things somewhat