25 — Convective Stability [*Revision* : 1.4]

- Bouyancy
  - Familiar concept: an object lighter than its surroundings rises (bubbles, hot air balloons)
  - Consequence of hydrostatic stratification
  - Consider bubble with density $\rho^{(\mathrm{b})}$ and volume $\mathrm{d}\tau^{(\mathrm{b})}$, immersed in surroundings with density $\rho^{(\mathrm{s})}$.
  - Force on bubble due to surrounding pressure gradients is

    $$F_{\mathrm{p}} = -\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{d}\tau^{(\mathrm{b})}$$

    Important Note: no $^{\mathrm{b}}$ or $^{\mathrm{s}}$ on P because pressure inside and outside bubble are same — pressure equalization is almost instantaneous (timescale is blob size / sound speed)
  - Assume surroundings are in hydrostatic equilibrium; pressure gradient is

    $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho^{(\mathrm{s})}g$$

    and so

    $$F_{\mathrm{p}} = \rho^{(\mathrm{s})}g\mathrm{d}\tau^{(\mathrm{b})}$$
  - Force on bubble due to gravity is

    $$F_{\mathrm{g}} = -g\rho^{(\mathrm{b})}\mathrm{d}\tau^{(\mathrm{b})}$$
  - Net force

    $$F = F_{\mathrm{p}} + F_{\mathrm{g}} = g(\rho^{(\mathrm{s})} - \rho^{(\mathrm{b})})\mathrm{d}\tau^{(\mathrm{b})},$$

    or per-unit-volume

    $$f = \frac{F}{\mathrm{d}\tau^{(\mathrm{b})}} = g(\rho^{(\mathrm{s})} - \rho^{(\mathrm{b})})$$
- Convective Instability
  - Consider bubble starting out at same density, temperature as surroundings; $f = 0$
  - Bubble then displaced Δr in radial direction (by random fluctuation); net force per unit volume is

    $$f = g(\Delta\rho^{(\mathrm{s})} - \Delta\rho^{(\mathrm{b})})$$

    where $\Delta\rho^{(\mathrm{s})}$ is change in density of surroundings, and $\Delta\rho^{(\mathrm{b})}$ is change in density of blob
  - If f has same sign as Δr, then force will push blob further away from starting position — **instability**!
  - Result of this instability is **convection**: rising and falling motions that transport energy
  - How do we calculate $\Delta\rho^{(\mathrm{s})}$ and $\Delta\rho^{(\mathrm{b})}$?
- Density Changes
  - Surrounding density change $\Delta\rho^{(\mathrm{s})}$ given by ambient density gradient:

    $$\Delta\rho^{(\mathrm{s})} = \frac{\mathrm{d}\rho}{\mathrm{d}r}\Delta r.$$
  - Bubble change depends on the thermodynamic processes going on in blob as it moves
  - For simplicity, assume no energy exchange with surroundings — adiabatic motion

- For adiabatic changes,

$$\frac{\Delta P}{P} = \gamma \frac{\Delta \rho}{\rho}$$

  where γ is ratio of specific heats of gas (see O&C, eqn. 10.81)
- So,

$$\Delta \rho^{(\mathrm{b})} = \frac{1}{\gamma} \frac{\rho}{P} \Delta P = \frac{1}{\gamma} \frac{\rho}{P} \frac{\mathrm{d}P}{\mathrm{d}r} \Delta r$$

- Bubble Motion
  - Putting together, net force per unit volume on bubble:

$$f = g \left(\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{1}{\gamma} \frac{\rho}{P} \frac{\mathrm{d}P}{\mathrm{d}r} \right) \Delta r$$

  - Apply Newton's Second Law to bubble:

$$\rho \frac{\mathrm{d}^2 \Delta r}{\mathrm{d}t^2} = -\rho N^2 \Delta r$$

    where

$$N^2 = g \left(\frac{1}{\gamma} \frac{\mathrm{d} \ln P}{\mathrm{d}r} - \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} \right)$$

    is the square of the **Brunt-Väisälä frequency**.
  - Solutions with $N^2 > 0$:

$$\Delta r \propto \begin{cases} \sin(Nt) \\ \cos(Nt) \end{cases}$$

    Oscillatory motion; **stable**, no convection
  - Solutions with $N^2 < 0$:

$$\Delta r \propto \begin{cases} \mathrm{e}^{\sqrt{|N^2|}t} \mathrm{e}^{-\sqrt{|N^2|}t} \end{cases}$$

    Run-away motion; **unstable**, convection

- Stability Criteria
  - Criterion for convective stability is $N^2 > 0$; this is the **Schwarzschild Criterion:**

$$\frac{1}{\gamma} \frac{\mathrm{d} \ln P}{\mathrm{d}r} > \frac{\mathrm{d} \ln \rho}{\mathrm{d}r}$$

  - More common form for Schwarzschild Criterion is in terms of temperature gradients.
  - For ideal gas, neglecting changes in μ,

$$\frac{\mathrm{d} \ln P}{\mathrm{d}r} = \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} + \frac{\mathrm{d} \ln T}{\mathrm{d}r}$$

    So, eliminating density gradient, stability criterion is

$$\frac{1}{\gamma} \frac{\mathrm{d} \ln P}{\mathrm{d}r} > \frac{\mathrm{d} \ln P}{\mathrm{d}r} - \frac{\mathrm{d} \ln T}{\mathrm{d}r}$$

    Simplfying:

$$\frac{\mathrm{d} \ln T}{\mathrm{d}r} > \frac{\gamma - 1}{\gamma} \frac{\mathrm{d} \ln P}{\mathrm{d}r}$$

- Dividing through by $-\mathrm{d}\ln P/\mathrm{d}r$,

$$\nabla > \nabla_{\mathrm{ad}}$$

where

$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln r}$$

is the **physical temperature gradient** — quantity that depends on temperature gradients in star; and

$$\nabla_{\mathrm{ad}} \equiv \frac{\gamma - 1}{\gamma}$$

is **adiabatic temperature gradient** — quantity that depends on thermodynamic properties of stellar material

- Convective & Radiative Regions
  - In any region, energy transport occurs by radiation alone (if convectively stable), or by combination of radiation and convection (if convectively unstable)
  - To find out what happens in a given region, consider radiative diffusion equation:

$$F_{\mathrm{rad}} = -\frac{4acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

Rewrite as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \nabla = -\frac{3\kappa\rho F_{\mathrm{rad}}}{4acT^4}\left(\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right)^{-1}.$$

  - Introduce pressure scale height as

$$H_P = -\left(\frac{\mathrm{d}\ln P}{\mathrm{d}r}\right)^{-1};$$

then,

$$\nabla = \frac{3\kappa\rho F_{\mathrm{rad}}}{4acT^4}H_P.$$

  - Guided by this expression, introduce **radiative temperature gradient** as

$$\nabla_{\mathrm{rad}} = \frac{3\kappa\rho F}{4acT^4}H_P,$$

where F is total flux; this corresponds to value of ∇ that would ensue *if* all energy transport is via radiation (i.e., if $F = F_{\mathrm{rad}}$)
  - Use radiative gradient to test for convection: if $\nabla_{\mathrm{rad}} > \nabla_{\mathrm{ad}}$, then region is **superadiabatic**, and will be convectively unstable
  - Once convective motions are established, the following inequalities hold:
    * $\nabla \geq \nabla_{\mathrm{ad}}$
    * $\nabla \leq \nabla_{\mathrm{rad}}$
  - Actual value of ∇ that ensues depends on details of convective energy transport