24 — Simple Radiative Models [*Revision* : 1.1]

- Recapitulation of stellar structure equations
  - Hydrostatic equilibrium
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g = -\rho \frac{GM_\mathrm{r}}{r^2}$$
  - Mass-radius
$$\frac{\mathrm{d}M_\mathrm{r}}{\mathrm{d}r} = 4\pi r^2 \rho$$
  - Energy conservation
$$\frac{\mathrm{d}L_\mathrm{r}}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon$$
  - Radiative diffusion
$$F_\mathrm{rad} = -\frac{1}{\bar{\kappa}\rho} \frac{\mathrm{d}P_\mathrm{rad}}{\mathrm{d}r} = -\frac{4acT^3}{3\bar{\kappa}\rho} \frac{\mathrm{d}T}{\mathrm{d}r}$$
    where $P_\mathrm{rad} = aT^4/3$ is radiation pressure
- Scaling relations for radiative models
  - Assume all energy is transported by radiation
  - Look for simple scaling relations between mass, radius and luminosity
  - Drop all numerical constants, assume $\bar{\kappa}$, ϵ are uniform in star
  - Mass-radius
$$\frac{M}{R} \sim R^2 \rho$$
    so
$$\rho \sim \frac{M}{R^3}$$
    density is mass divided by characteristic volume
  - Hydrostatic equilibrium & mass-radius
$$\frac{P}{R} \sim \frac{M}{R^3} \frac{GM}{R^2}$$
    so
$$P \sim \frac{GM^2}{R^2} \frac{1}{R^2}$$
    pressure is charateristic gravitational force divided by characteristic area
  - Energy conservation & mass-radius
$$\frac{L}{R} \sim R^2 \rho \epsilon$$
    so
$$L \sim R^3 \rho \epsilon \sim M\epsilon$$
    luminosity is mass times energy generation rate per unit mass
  - Radiative diffusion & mass-radius
$$F_\mathrm{rad} \sim \frac{L}{R^2} \sim \frac{acT^3}{\bar{\kappa}\rho} \frac{T}{R}$$
    so
$$L \sim \left(aT^4\right) \; R^3 \; \left(\frac{Rc}{\bar{\kappa}M}\right)$$
    luminosity is radiant energy per unit volume times characteristic volume divided by diffusion timescale t_diff

    * Diffusion timescale for any process is $t_{\text{diff}} \sim R^2/D$, where R is characteristic length, and D is diffusion coefficient
    * $D \sim v\langle s\rangle$, where v is velocity and $\langle s\rangle$ is mean free path
    * For photons diffusing in star $v \sim c$ and $\langle s\rangle \sim 1/\bar{\kappa}\rho \sim R^3/\bar{\kappa}M$ so $D \sim cR^3/\bar{\kappa}M$
    * Hence, $t_{\text{diff}} \sim \bar{\kappa}M/Rc$
  - Final relation required to close above equations is equation-of-state
    * Ideal gas
$$P \sim \frac{\rho kT}{\mu m_{\rm H}} \sim \frac{kMT}{\mu m_{\rm H} R^3}$$
    * Radiation
$$P \sim aT^4$$
  - Focus on ideal gas case
    * Eliminate pressure using hydrostatic and EOS
$$\frac{kMT}{\mu m_{\rm H} R^3} \sim \frac{GM^2}{R^4}$$
      so
$$TR \sim \frac{GM\mu m_{\rm H}}{k}$$
    * Eliminate $(TR)^4$ from radiative diffusion
$$L \sim a\left(\frac{GM\mu m_{\rm H}}{k}\right)^4 \frac{c}{\bar{\kappa}M}$$
      so
$$L \sim \frac{ac}{\bar{\kappa}}\left(\frac{G\mu m_{\rm H}}{k}\right)^4 M^3$$
    * Big result: standard mass-luminosity relation $L \sim M^3$
    * Radius does not enter into this relation
    * Note dependence on μ^4 — stars with higher molecular weight are more luminous for the same mass

- Eddington standard model
  - Slightly more accurate model for a radiative star
  - Assume that all mass, energy generation concentrated right at center of star; so, away from center
$$M_{\rm r} = M$$
    and
$$L_{\rm r} = L$$
  - Also assume that opacity is constant
  - Radiative diffusion equation can now be written as
$$\frac{{\rm d}P_{\rm rad}}{{\rm d}r} = -\rho\frac{\bar{\kappa}L}{4\pi cr^2}$$
  - Compare against hydrostatic equilibrium
$$\frac{{\rm d}P}{{\rm d}r} = -\rho\frac{GM}{r^2}$$

- Take ratio
$$\frac{\mathrm{d}P}{\mathrm{d}P_{\mathrm{rad}}} = \frac{4\pi GMc}{\bar{\kappa}L}$$
where right-hand side is *constant*
- In fact, this constant is related to Eddington parameter:
$$\Gamma = \frac{\bar{\kappa}L}{4\pi GMc}$$
- So,
$$\frac{\mathrm{d}P}{\mathrm{d}P_{\mathrm{rad}}} = \frac{1}{\Gamma},$$
and so
$$P = \frac{P_{\mathrm{rad}}}{\Gamma}$$
(set constant of integration to zero).
- Define ratio of gas pressure to total pressure:
$$P_{\mathrm{g}} = \beta P$$
so
$$P_{\mathrm{rad}} = (1-\beta)P$$
- Because $P \propto P_{\mathrm{rad}}$ in Eddington standard model, see that β is constant throughout, and equal to $1-\Gamma$
- Use EOS to find pressure-density relation:
$$P = \frac{1}{1-\beta}\frac{aT^4}{3}$$
but
$$T = \frac{P_{\mathrm{g}}\mu m_{\mathrm{H}}}{\rho k}$$
so
$$P = \frac{1}{1-\beta}\frac{a}{3}\left(\frac{P_{\mathrm{g}}\mu m_{\mathrm{H}}}{\rho k}\right)^4$$
Resulting scaling:
$$P \sim \rho^{4/3}$$
- One-to-one relationship between total pressure means Eddington standard model is a $\gamma = 4/3$ ($n=3$) polytrope