22 — Radiative Diffusion $[Revision : 1.1]$

- Interior radiation field
  - Deep inside star, temperature gradients over one mean free path are v. small
  - Material behaves as if it were in isothermal enclosure — almost (but not completely) isotropic
  - Approximate specific intensity using first-order expansion

$$I(r + \mathrm{d}r, \mu) \approx I(r, \mu) + I'(r, \mu)\mathrm{d}r$$

    (v. accurate in stellar interior)
  - Radiation will obey Eddington approximation (see Assignment 3, Q1):

$$P_{\mathrm{rad}} = \frac{4\pi}{3c}\langle I \rangle$$

    Leads to two useful results...

- Radiation pressure
  - Recall (Notes 9): mean intensity related to radiation energy density:

$$u = \frac{4\pi}{c}\langle I \rangle$$

  - But for matter in thermal equilibrium at temperature T,

$$u = aT^4$$

  - Hence

$$\langle I \rangle = \frac{acT^4}{4\pi}$$

    and

$$P_{\mathrm{rad}} = \frac{aT^4}{3}$$

  - The radiation pressure in the latter equation must be added to ideal gas equation of state (Notes 21), to get expression for total gas pressure:

$$P = P_{\mathrm{g}} + P_{\mathrm{rad}} = \frac{\rho k T}{\mu m_{\mathrm{H}}} + \frac{aT^4}{3}$$

- Radiative flux
  - From first moment of the radiative transfer equation for *gray* medium:

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}\tau} = \frac{F_{\mathrm{rad}}}{c}$$

    (now writing flux as F_{rad} to distinguish from other ways to transport energy in interior)
  - With $\mathrm{d}\tau = \kappa\rho\mathrm{d}r$,

$$-\frac{1}{\kappa\rho}\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = \frac{F_{\mathrm{rad}}}{c}$$

  - Rearrange, and use previous expression for P_{rad}:

$$F_{\mathrm{rad}} = -\frac{c}{\kappa\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{aT^4}{3}\right) = -\frac{4ac}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

- This is the **radiative diffusion equation** — it tells us the temperature gradient necessary to push a radiative energy flux F_{rad} through the stellar material
- Similar equation obeyed by **conduction**; but generally unimportant

- Rosseland mean opacity
  - In fact, material in interior cannot be treated as gray
  - But similar radiative diffusion equation can be derived

$$F_{\mathrm{rad}} = - - \frac{4ac}{3\bar{\kappa}\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

  - Now, $\bar{\kappa}$ is **Rosseland mean opacity**, defined by

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\lambda}\frac{\mathrm{d}B_\lambda}{\mathrm{d}T}\,\mathrm{d}\lambda}{\int_0^\infty \frac{\mathrm{d}B_\lambda}{\mathrm{d}T}\,\mathrm{d}\lambda}$$

    where $B_\lambda(T)$ is usual Planck function. This is a **harmonic mean** of opacity κ_λ, weighted by the temperature derivative of Planck function

- Hydrostatic equilibrium revisited
  - Recall equation of **hydrostatic equilibrium**:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

  - Insert above expression for total pressure P:

$$\frac{\mathrm{d}P_{\mathrm{g}}}{\mathrm{d}r} + \frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\rho g$$

  - From derivation of diffusion equation, eliminate radiation pressure gradient:

$$\frac{\mathrm{d}P_{\mathrm{g}}}{\mathrm{d}r} - \bar{\kappa}\rho\frac{F_{\mathrm{rad}}}{c} = -\rho g$$

  - Rearrange:

$$\frac{\mathrm{d}P_{\mathrm{g}}}{\mathrm{d}r} = -\rho g\left(1 - \frac{\kappa F_{\mathrm{rad}}}{gc}\right)$$

  - Quantity in brackets must be positive; so, there is limit to flux/luminosity in star
  - At surface, this **Eddington limit** is

$$\Gamma \equiv \frac{\kappa F_{\mathrm{rad}}}{gc} = \frac{\bar{\kappa}L}{4\pi GMc} < 1$$

    (used $L = 4\pi R^2 F_{\mathrm{rad}}$ at surface, since all energy is transported by radiation there).
  - Physical meaning: if $\Gamma > 1$, then radiation pressure overwhelms gravity, and radiation-driven mass loss from star must ensue