# 22 — Radiative Diffusion [*Revision* : 1.1]

- Interior radiation field
  - Deep inside star, temperature gradients over one mean free path are v. small
  - Material behaves as if it were in isothermal enclosure — almost (but not completely) isotropic
  - Approximate specific intensity using first-order expansion

    $$I(r + \mathrm{d}r, \mu) \approx I(r, \mu) + I'(r, \mu)\mathrm{d}r$$

    (v. accurate in stellar interior)
  - Radiation will obey Eddington approximation (see Assignment 3, Q1):

    $$P_{\mathrm{r}} = \frac{4\pi}{3c}\langle I \rangle$$

    Leads to two useful results...
- Radiation pressure
  - Recall (Notes 9): mean intensity related to radiation energy density:

    $$u = \frac{4\pi}{c}\langle I \rangle$$

  - But for matter in thermal equilibrium at temperature $T$,

    $$u = aT^4$$

  - Hence

    $$\langle I \rangle = \frac{acT^4}{4\pi}$$

    and

    $$P_{\mathrm{r}} = \frac{aT^4}{3}$$

  - The radiation pressure in the latter equation must be added to ideal gas equation of state (Notes 21), to get expression for total gas pressure:

    $$P = P_{\mathrm{g}} + P_{\mathrm{r}} = \frac{\rho k T}{\mu m_{\mathrm{H}}} + \frac{aT^4}{3}$$

- Radiative flux
  - From first moment of the radiative transfer equation for *gray* medium:

    $$\frac{\mathrm{d}P_{\mathrm{r}}}{\mathrm{d}\tau} = \frac{F_{\mathrm{r}}}{c}$$

    (now writing flux as $F_{\mathrm{r}}$ to distinguish from other ways to transport energy in interior)
  - With $\mathrm{d}\tau = \kappa\rho\mathrm{d}r$,

    $$-\frac{1}{\kappa\rho}\frac{\mathrm{d}P_{\mathrm{r}}}{\mathrm{d}r} = \frac{F_{\mathrm{r}}}{c}$$

  - Rearrange, and use previous expression for $P_{\mathrm{r}}$:

    $$F_{\mathrm{r}} = -\frac{c}{\kappa\rho}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{aT^4}{3}\right) = -\frac{4ac}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

- This is the **radiative diffusion equation** — it tells us the temperature gradient necessary to push a radiative energy flux $F_\mathrm{r}$ through the stellar material

- Rosseland mean opacity
  - In fact, material in interior cannot be treated as gray
  - But similar radiative diffusion equation can be derived

$$F_\mathrm{r} = - - \frac{4ac}{3\bar{\kappa}\rho} \frac{\mathrm{d}T}{\mathrm{d}r}$$

  - Now, $\bar{\kappa}$ is **Rosseland mean opacity**, defined by

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\lambda} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda}{\int_0^\infty \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda}$$

    where $B_\lambda(T)$ is usual Planck function. This is a **harmonic mean** of opacity $\kappa_\lambda$, weighted by the temperature derivative of Planck function

- Hydrostatic equilibrium revisited
  - Recall equation of **hydrostatic equilibrium**:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$

  - Insert above expression for total pressure $P$:

$$\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}r} + \frac{\mathrm{d}P_\mathrm{r}}{\mathrm{d}r} = -\rho g$$

  - From derivation of diffusion equation, eliminate radiation pressure gradient:

$$\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}r} - \bar{\kappa}\rho\frac{F_\mathrm{r}}{c} = -\rho g$$

  - Rearrange:

$$\frac{\mathrm{d}P_\mathrm{g}}{\mathrm{d}r} = -\rho g \left(1 - \frac{\kappa F_\mathrm{r}}{gc}\right)$$

  - Quantity in brackets must be positive; so, there is limit to flux/luminosity in star
  - At surface, this *Eddington limit* is

$$\Gamma \equiv \frac{\kappa F_\mathrm{r}}{gc} = \frac{\bar{\kappa} L}{4\pi G M c} < 1$$

    (used $L = 4\pi R^2 F_\mathrm{r}$ at surface, since all energy is transported by radiation there).
  - Physical meaning: if $\Gamma > 1$, then radiation pressure overwhelms gravity, and radiaiton-driven mass loss from star must ensue