20 — Polytropes $[Revision : 1.1]$

- Mechanical Equations
  - So far, two differential equations for stellar structure:
    - **hydrostatic equilibrium**:
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{GM_r}{r^2}$$
    - **mass-radius relation**:
$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho$$
  - Two equations involve three unknowns: pressure P, density ρ, mass variable M_r — cannot solve
  - Try to relate P and ρ using a gas equation of state — e.g., ideal gas:
$$P = \frac{\rho k T}{\mu}$$
    ...but this introduces extra unknown: temperature T
  - To eliminate T, must consider **energy transport**
- Polytropic Equation-of-State
  - Alternative to having to do full energy transport
  - Used historically to create simple stellar models
  - Assume some process means that pressure and density always related by **polytropic equation of state**
$$P = K\rho^\gamma$$
    for constant K, γ
  - Polytropic EOS resembles pressure-density relation for adibatic change; but γ is not necessarily equal to usual ratio of specific heats
  - Physically, gases that follow polytropic EOS are either
    - Degenerate — Fermi-Dirac statistics apply (e.g., non-relativistic degenerate gas has $P \propto \rho^{5/3}$)
    - Have some 'hand-wavy' energy transport process that somehow maintains a one-to-one pressure-density relation
- Polytropes
  - A **polytrope** is simplified stellar model constructed using polytropic EOS
  - To build such a model, first write down hydrostatic equilibrium equation in terms of gradient of graqvitational potential:
$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$
  - Eliminate pressure using polytropic EOS:
$$\frac{K}{\rho} \frac{\mathrm{d}\rho^\gamma}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

- Rearrange:
$$\frac{K\gamma}{\gamma-1}\frac{\mathrm{d}\rho^{\gamma-1}}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$
- Solving:
$$K(n+1)\rho^{1/n} = -\Phi$$
where $n \equiv 1/(\gamma-1)$ is the **polytropic index** (do not confuse with number of degrees of freedom — **different!**). Note that constant of integration chosen so that surface $\rho \to 0$ corresponds to $\Phi \to 0$)
- Significance of result: density and gravitational potential are directly related to one another
- To progress further, use other relation between density and potential — **Poisson's equation**:
$$\nabla^2\Phi = 4\pi G\rho;$$
in spherical symmetry,
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho$$
- Combining:
$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{1/n-1}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho$$
- Messy equation; but can reduce to a dimensionless form by writing density as
$$\rho(r) = \rho_{\mathrm{c}}\left[D_n(r)\right]^n$$
where ρ_{c} is central density. Also, write radius as
$$r = \lambda_n\xi$$
where
$$\lambda_n = \left[(n+1)\left(\frac{K\rho_{\mathrm{c}}^{1/n-1}}{4\pi G}\right)\right]^{1/2}$$
- Then, equation becomes
$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi^2\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right] = -D_n^n$$
which is **Lane-Emden equation**

- Solving the LE equation
  - To solve equation, need boundary conditions
  - Because $\rho \to \rho_{\mathrm{c}}$ at center,
$$D_n|_{\xi=0} = 1$$
  - Also, mass continuity requires that
$$\left.\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right|_{\xi=0} = 0$$
  - At surface $\xi = \xi_1$, density goes to zero
$$D_n|_{\xi=\xi_1} = 0$$
(in fact, this equation serves to define ξ_1). Radius of star follows as
$$R = \lambda_n\xi_1$$

- Integrate over density to find mass of star:

$$M = \int_0^R 4\pi r^2 \rho \, \mathrm{d}r = -4\pi \lambda_n^3 \rho_\mathrm{c} \xi_1^2 \left. \frac{\mathrm{d}D_n}{\mathrm{d}\xi} \right|_{\xi=\xi_1}.$$

- In practice, to construct a polytropic stellar model, first choose M, R and n. After solving LE equation, find λ_n from radius relation above; find ρ_c from mass relation; and find K from definition of λ_n.

- Properties of LE solutions
  - Analytic only for $n = 0$, $n = 1$ and $n = 5$
  - For $n \geq 5$, ξ_1 is infinite (and so radius is infinite)
  - Special cases:
    * Homogeneous (constant density) : $n = 0$
    * Isothermal (constant temperature) : $n \to \infty$