19 — Stellar Energy [$Revision: 1.4$]

- Gravitational energy
  - To calculate gravitational energy E_{g} of star, imagine building up star by sequentially adding shells
  - Adding on shell of mass $\mathrm{d}m$ changes gravitational energy by

$$\mathrm{d}E_{\mathrm{g}} = \Phi\,\mathrm{d}m$$

    where Φ is gravitational potential
  - If shell has radius r, density ρ and thickness $\mathrm{d}r$,

$$\mathrm{d}m = 4\pi r^2 \rho\,\mathrm{d}r$$

  - If no matter outside r,

$$\Phi = -\frac{GM_r}{r}$$

  - Putting together

$$\mathrm{d}E_{\mathrm{g}} = -\frac{GM_r}{r} 4\pi r^2 \rho\,\mathrm{d}r$$

  - Adding up contribution of all shells from $r = 0$ to $r = R$,

$$E_{\mathrm{g}} = \int \mathrm{d}E_{\mathrm{g}} = -\int_0^R \frac{GM_r}{r} 4\pi r^2 \rho\,\mathrm{d}r$$

- Thermal energy
  - To calculate thermal energy of star, add up thermal energy per unit volume u:

$$E_{\mathrm{t}} = \int_0^R u 4\pi r^2\,\mathrm{d}r$$

  - For ideal gas, u related to pressure:

$$u = \frac{P}{\gamma - 1}$$

    where γ is ratio of specific heats
  - So,

$$E_{\mathrm{t}} = \int_0^R \frac{P}{\gamma - 1} 4\pi r^2\,\mathrm{d}r$$

- Virial theorem
  - Consider star in hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho = -\frac{GM_r}{r^2}\rho$$

  - Multiply both sides by $4\pi r^3$, integrate over all radii:

$$\int_0^R \frac{\mathrm{d}P}{\mathrm{d}r} 4\pi r^3\,\mathrm{d}r = -\int_0^R \frac{GM_r}{r^2} 4\pi r^3 \rho\,\mathrm{d}r = E_{\mathrm{g}}$$

- Do integral on lhs by parts:

$$\left[4\pi r^3 P\right]_0^R - 3\int_0^R P 4\pi r^2 \, \mathrm{d}r = E_\mathrm{g}$$

- Assuming $P \to 0$ at $r \to R$,

$$-3\int_0^R P 4\pi r^2 \, \mathrm{d}r = E_\mathrm{g}$$

- Rewrite lhs in terms of thermal energy:

$$-3(\gamma - 1)E_\mathrm{t} = E_\mathrm{g}$$

- This is the **virial theorem** — it relates the thermal and gravitational energy of any self-gravitating system in hydrostatic equilibrium
- Special case: monatomic gas, $\gamma = 5/3$:

$$E_\mathrm{g} = -2E_\mathrm{t}$$

- Total energy
  - Define **total energy** of star as sum of thermal, gravitational and nuclear energies:

  $$E = E_\mathrm{t} + E_\mathrm{g} + E_\mathrm{n}$$

  - For the moment, assume $E_\mathrm{n} = 0$, and apply virial theorem:

  $$E = -E_\mathrm{t} = \frac{E_\mathrm{g}}{2}$$

  - Time derivative of E must equal energy lost by star — luminosity:

  $$L = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E_\mathrm{t}}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}E_\mathrm{g}}{\mathrm{d}t}$$

  - Because L is positive, E_t must get more positive with time (star heats up), and E_g must get more negative (star shrinks).
  - This is **Kelvin-Helmholtz contraction**: in a star with no other energy sources, luminosity is supplied by gravitational energy release (half goes into luminosity, other half goes into thermal energy)
  - Typical timescale of KH contraction:

  $$t_\mathrm{KH} \sim \frac{|E_\mathrm{g}|}{L} = \frac{GM^2}{LR}$$

  - For Sun, $t_\mathrm{KH} \sim 30\,\mathrm{Myr}$, much shorter than age of Earth or solar system. So, the Sun cannot be powered by KH contraction
  - In fact, Sun's energy comes from nuclear energy, with no contraction:

  $$L = -\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}E_\mathrm{n}}{\mathrm{d}t}$$

  - However, KH contraction important for pre-main-sequence stars:
    - Initially, star too cold/low density to have nuclear reactions in core
    - KH contraction causes star to shrink and heat up
    - Eventually, nuclear reactions begin in core
    - With further contraction/heating, reactions become efficient enough to provide star's luminosity
    - Then, contraction stops — star is now at zero-age main sequence (ZAMS)