18 — Mechanical Structure [$Revision$: 1.2]

- Focus now changes from the surface, atmosphere layers of a star, over to the interior layers: **stellar structure**
- **Hydrostatic equilibrium** is one of the fundamental principles of stellar structure.
- Along the same lines as derivation in Notes 13:
  - Star supports itself against downward pull of gravity by **pressure gradients**
  - Consider slab inside star, extending from radius r to radius $r + \mathrm{d}r$ (i.e., thickness is $\mathrm{d}r$), and with horizontal area $\mathrm{d}A$
    * Upward pressure force on inner side is $P(r)\,\mathrm{d}A$
    * Downward pressure force on upper side is $P(r + \mathrm{d}r)\,\mathrm{d}A$
    * Net pressure force is
    $$F_P = P(r)\,\mathrm{d}A - P(r + \mathrm{d}r)\,\mathrm{d}A$$
    * Net gravitational force is likewise
    $$F_g = -g\rho\,\mathrm{d}r\,\mathrm{d}A$$
    * For static balance:
    $$F_P + F_g = P(r)\,\mathrm{d}A - P(r + \mathrm{d}r)\,\mathrm{d}A - g\rho\,\mathrm{d}r\,\mathrm{d}A = 0$$
    * Divide through by $\mathrm{d}r, \mathrm{d}A$, take limit $\mathrm{d}r \to 0$:
    $$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho g$$
    This is **equation of hydrostatic equilibrium** — relates pressure gradient to local density and gravitational acceleration
- Gravitational field
  - Obtain g from gravitational potential Φ:
  $$g = \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$
  - Gravitational potential comes from solution of **Poisson's equation**:
  $$\nabla^2\phi = 4\pi G\rho$$
  - Special-case solution: in **spherical symmetry**,
  $$g = \frac{GM_r}{r^2}$$
  where M_r is total mass contained by the sphere with radius r
  - Important: note that expression for g does **not** imply that $\Phi = -GM_r/r$ (applies only when there is no matter outside r)
- Mass distribution
  - To find M_r, just add up mass contained within sphere:
  $$M_r = \int_0^r 4\pi r'^2 \rho(r')\,\mathrm{d}r'$$

  - M_r varies between 0 (core) and M (surface)
  - Often, useful to use M_r instead of r as coordinate for describing position within star

- Timescales
  - Suppose pressure could be turned off. Apply Newton's second law to find approximate timescale for stellar collapse:
$$M\frac{R}{t^2} \sim \frac{GM^2}{R^2}$$
    Solving, **freefall timescale** is
$$t_{\rm ff} = \sqrt{\frac{R^3}{GM}}$$
  - Likewise, suppose gravity could be turned off. Apply Newton's second law to find approximate timescale for stellar explosion:
$$M\frac{R}{t^2} \sim PR^2$$
    Solving, **explosion timescale** is
$$t_{\rm ex} = \sqrt{\frac{M}{PR}}$$
  - For star in hydrostatic equilibrium,
$$\frac{P}{R} \sim \frac{GM}{R^2}\frac{M}{R^3}$$
    (first term on rhs is gravity, second term is density), and so
$$t_{\rm ex} = \sqrt{\frac{M}{R^2}\frac{R}{P}} = \sqrt{\frac{R^3}{GM}} = t_{\rm ff}$$
    (i.e., freefall and explosion timescales are equal)
  - **Dynamical timescale**: single timescale for star in hydrostatic equilibrium:
$$t_{\rm dyn} = t_{\rm ex} = t_{\rm ff} = \sqrt{\frac{R^3}{GM}}$$
    Dynamical timescale is typically ~ 1 hour for main-sequence star; indicates how long it will take for star to correct any departures from hydrostatic equilibrium