13 — The Gray & Eddington Approximations [*Revision* : 1.3]

- Gray atmospheres
  - Plane-parallel RTE still difficult to solve because vertical optical depth is wavelength dependent
  - Useful simplification of **gray atmosphere**: assume opacity is independent of wavelength: $\kappa_\lambda = \bar{\kappa}$. Then,

    $$\tau_{\rm v}(z) = \int_z^0 \bar{\kappa}(z)\rho(z)\mathrm{d}z$$

    and

    $$\cos\theta \frac{\mathrm{d}I}{\mathrm{d}\tau_{\rm v}} = I - S$$

    where

    $$I = \int_0^\infty I_\lambda \mathrm{d}\lambda, \qquad S = \int_0^\infty S_\lambda \mathrm{d}\lambda$$

- Moments of the radiation field
  - Grey, plane-parallel RTE still depends on direction μ
  - To further simplify, integrate RTE over all solid angles:

    $$\frac{\mathrm{d}}{\mathrm{d}\tau_{\rm v}} \int I\mu \mathrm{d}\Omega = \int I \mathrm{d}\Omega - S \int \mathrm{d}\Omega$$

    (S comes out from under integral sign, because it doesn't depend on angle)
  - Recalling from notes 9, mean intensity:

    $$\langle I \rangle = \frac{1}{4\pi} \int I \mathrm{d}\Omega = \frac{1}{2} \int_{-1}^{1} I \mathrm{d}\mu,$$

    and flux:

    $$F = \int I\mu \mathrm{d}\Omega = 2\pi \int_{-1}^{1} I\mu \mathrm{d}\mu$$

  - So,

    $$\frac{\mathrm{d}F}{\mathrm{d}\tau_{\rm v}} = 4\pi(\langle I \rangle - S)$$

  - Another equation is obtained by multiplying RTE by μ, and integrating over all solid angles:

    $$\frac{\mathrm{d}}{\mathrm{d}\tau_{\rm v}} \int I\mu^2 \mathrm{d}\Omega = \int I\mu \mathrm{d}\Omega - S \int \mu \mathrm{d}\Omega$$

  - Recalling from notes 9, radiation pressure:

    $$P_{\rm rad} = \frac{1}{c} \int I\mu^2 \mathrm{d}\Omega = \frac{2\pi}{c} \int_{-1}^{1} I\mu^2 \mathrm{d}\mu$$

    and so:

    $$\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}\tau_{\rm v}} = \frac{F}{c}$$

  - Equations for $\mathrm{d}F/\mathrm{d}\tau_{\rm v}$ and $\mathrm{d}P_{\rm rad}/\mathrm{d}\tau_{\rm v}$ are **moment equations**; $n^{\rm th}$ moment of radiation field defined by

    $$M_n = \int I\mu^n \mathrm{d}\Omega$$

- Radiative equilibrium
  - In **radiative equilibrium**, no net energy added or subtracted as radiation travels up through atmosphere — amount absorbed = amount emitted:

$$\kappa\langle I\rangle = \kappa S$$

  - From first moment equation, this means that

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = 4\pi(\langle I\rangle - S) = 0$$

    i.e., flux F is constant throughout atmosphere
  - From second moment equation, with F constant:

$$P_{\mathrm{rad}} = \frac{1}{c}F\tau_{\mathrm{v}} + C$$

    where C is constant of integration
  - **However**, how do we find out P_{rad}? Could take next higher-order moment of RTE, but that will just introduce an additional term M_3 — and so on, *ad infinitum*
  - In fact, we need **closure relation**: independent piece of information that will relate P_{rad} to $\langle I\rangle$
- Eddington approximation
  - To find closure relation, make simplifying **Eddington approximation** about radiation field: specific intensity is constant over upward and downward hemispheres:

$$I(\mu) = \begin{cases} I_{\mathrm{out}} & \mu > 0 \\ I_{\mathrm{in}} & \mu < 0 \end{cases}$$

  - Substitute into definitions above for various moments gives

$$\langle I\rangle = \frac{1}{2}(I_{\mathrm{out}} + I_{\mathrm{in}})$$
$$F = \pi(I_{\mathrm{out}} - I_{\mathrm{in}})$$
$$P_{\mathrm{rad}} = \frac{2\pi}{3c}(I_{\mathrm{out}} + I_{\mathrm{in}}) = \frac{4\pi}{3c}\langle I\rangle$$

  - Last equation gives the desired closure relation — an independent piece of information relating radiation pressure to mean intensity
  - Combining with earlier equation for radiation pressure:

$$\frac{4\pi}{3c}\langle I\rangle = \frac{1}{c}F\tau_{\mathrm{v}} + C$$

  - Eliminate C by requiring that $I_{\mathrm{in}} = 0$ at τ_{v}=0:

$$\frac{4\pi}{3}\langle I\rangle = F\left(\tau_{\mathrm{v}} + \frac{2}{3}\right)$$

  - Final step: since F is constant in atmosphere, and at outer layers $F = \sigma T_{\mathrm{eff}}^4$ (**definition** of effective temperature!),

$$\langle I\rangle = \frac{3\sigma}{4\pi}T_{\mathrm{eff}}^4\left(\tau_{\mathrm{v}} + \frac{2}{3}\right)$$

    This then gives source function throughout atmosphere, and hence full radiation field via formal solution

– If atmosphere is in **local thermodynamic equilibrium**, $S = B = \sigma T^4/\pi$, and so we have **temperature structure**:

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau_{\text{v}} + \frac{2}{3} \right)$$