Assignment 5 — Solutions [Revision : 1.2]

- 1. See Figs. 1 and 2 for the plots of the ∇ 's as a function of $\log T$.
 - (a) Zones where convection is taking place have $\nabla_{rad} > \nabla_{ad}$.
 - (b) Convection is efficient when $\nabla \approx \nabla_{ad}$ (i.e., the temperature gradient is close to adiabatic); it is inefficient when ∇ is appreciably larger than ∇_{ad} (i.e., the temperature gradient is superadiabatic).

[6 points]

2. Applying the Saha equation to hydrogen, with the given partition functions, gives the ratio between the number of ionized hydrogen atoms (N_2) and the number of neutral hydrogen atoms (N_1) as

$$\frac{N_2}{N_1} = \frac{1}{n_{\rm e}} \left(\frac{2\pi m_{\rm e} kT}{h^2}\right)^{3/2} e^{-13.6 \, {\rm eV}/kT}$$

The ionization fraction is calculated from this ratio as

$$x = \frac{N_2}{N_1 + N_2} = \frac{1}{n_e \left(\frac{2\pi m_e kT}{h^2}\right)^{-3/2} e^{13.6 \text{ eV}/kT} + 1}$$

[3 points]

Fig. 3 shows the plot of x as a function of $\log T$. The corresponding values determined from columns 25 and 27 are shown. Deep in the star, the latter values stay close to unity, indicating full ionization; but those calculated using the Saha equation begin to fall, indicating some recombination.

[4 points]

This recombination is an artifact introduced by the Saha equation, which doesn't take into account the fact that — at the high densities found deep in the star — the ionization potential of each hydrogen atom is reduced below 13.6 eV by the overlapping electrostatic potential wells of neighboring atoms. When this reduction is taken into account, no recombination occurs.

[2 points]

3. Fig. 4 shows the plot of $\log \kappa$ as a function of $\log T$. Fig. 5 likewise shows the He III ionization fraction y_2 , obtained from columns 28 and 30 of the *EZ Web* file. The steep increase in y_2 around $\log T \approx 4.6$, seen in the latter figure, highlights the region where He II is being ionized into He III, and serves to identify which opacity peak in Fig. 4 is associated with helium. The other strong peak, at $\log T \approx 5.3$, therefore is the 'iron bump'. (A weaker peak at $\log T \approx 6.3$ is the 'deep iron bump'.)

[6 points]

4. Fig. 6 shows the plot of $\log \epsilon$ as a function of $\log T$. Also shown are the energy generation rates due to the PP chain and CNO cycle. The energy production by the PP chain exceeds that by the CNO cycle below $\log T \approx 7.15$, corresponding to $T \approx 14 \times 10^6$ K.

[4 points]

5. Fig. 7 shows the plot of Γ (the Eddington factor) as a function of log T. The star is above the Eddington limit around log $T \approx 5.3$, due to the opacity associated with the iron bump. To demonstrate how the star is able to remain in balance, Fig. 8 shows a plot of log P_{gas} as a function of log T. In the region around log $T \approx 5.3$, the pressure increases in the outward direction — i.e., there is a gas pressure inversion. The positive gas pressure gradient in this



Figure 1: The various temperature gradients plotted as a function of $\log T$, for the $1 M_{\odot}$ model. The solid line shows the physical gradient ∇ ; the dotted line shows the adiabatic gradient $\nabla_{\rm ad}$; and the dashed line shows the radiative gradient $\nabla_{\rm rad}$. The horizontal line under the plot indicate where convection is taking place, and whether it is efficient (thick) or inefficient (thin). (Q1)

region provides an inward force that counteracts the outward radiative force, and keeps the star from blowing itself apart.

[6 points]



Figure 2: The various temperature gradients plotted as a function of log T, for the $10 M_{\odot}$ model. The solid line shows the physical gradient ∇ ; the dotted line shows the adiabatic gradient $\nabla_{\rm ad}$; and the dashed line shows the radiative gradient $\nabla_{\rm rad}$. The horizontal lines under the plot indicate where convection is taking place, and whether it is efficient (thick) or inefficient (thin). (Q1)



Figure 3: The ionization fraction x of hydrogen, plotted as a function of log T for the $1 M_{\odot}$ model. The solid line indicates values calculated using the Saha equation, and the dashed line the values that come directly from the model. (Q2)



Figure 4: The opacity $\log \kappa$, plotted as a function of $\log T$ for the $10 M_{\odot}$ model. The arrows label the position of the peaks associated with helium and iron/nickel. (Q3)



Figure 5: The fraction y_2 of helium atoms that are in the twice-ionized state, plotted as a function of log T for the 10 M_{\odot} model. Note the sharp rise around log $T \approx 4.6$. (Q3)



Figure 6: The nuclear energy generation rate $\log \epsilon$, plotted as a function of $\log T$ for the $10 M_{\odot}$ model. The solid line shows the total rate; the dotted line the rate associated with the PP chain; and the dashed line the rate associated with the CNO cycle. (Q4)



Figure 7: The Eddington factor Γ , plotted as a function of log T for the 100 M_{\odot} model. This factor exceeds unity around log $T \approx 5.3$, the location of the iron-bump opacity peak. (Q5)



Figure 8: The gas pressure $\log P_{\text{gas}}$, plotted as a function of $\log T$ for the 100 M_{\odot} model. Note the pressure inversion around $\log T \approx 5.3$, which is responsible for balancing the outward radiative force. (Q1)