Assignment 4 — Solutions [*Revision* : 1.2]

1. (a) Rearranging the first-moment equation,

$$F_\lambda = -\frac{c}{\kappa_\lambda \rho} \frac{\mathrm{d}P_{\mathrm{rad},\lambda}}{\mathrm{d}r}$$

Substituting in the expression for $P_{\mathrm{rad},\lambda}$, this becomes

$$F_\lambda = -\frac{4\pi}{3\kappa_\lambda \rho} \frac{\mathrm{d}B_\lambda}{\mathrm{d}r}.$$

The chain rule is used to write

$$\frac{\mathrm{d}B_\lambda}{\mathrm{d}r} = \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}r},$$

from which we obtain the result

$$F_\lambda = -\frac{4\pi}{3\kappa_\lambda \rho} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}r}.$$

(b) Integrating over all wavelengths,

$$F = \int_0^\infty F_\lambda \,\mathrm{d}\lambda = -\int_0^\infty \frac{4\pi}{3\kappa_\lambda \rho} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \frac{\mathrm{d}T}{\mathrm{d}r} \,\mathrm{d}\lambda.$$

Bringing the wavelength-independent terms out from under the integral sign,

$$F = -\frac{4\pi}{3\rho} \frac{\mathrm{d}T}{\mathrm{d}r} \int_0^\infty \frac{1}{\kappa_\lambda} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda.$$

(c) Equating the two expressions for F,

$$\frac{4\pi}{3\rho} \frac{\mathrm{d}T}{\mathrm{d}r} \int_0^\infty \frac{1}{\kappa_\lambda} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda = \frac{4acT^3}{3\bar{\kappa}\rho} \frac{\mathrm{d}T}{\mathrm{d}r}.$$

Rearranging,

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\lambda} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda}{acT^3/\pi}.$$

Recognizing that

$$\int_0^\infty B_\lambda \,\mathrm{d}\lambda = \frac{\sigma T^4}{\pi} = \frac{acT^4}{4\pi},$$

the expression for $\bar{\kappa}$ may also be written

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\lambda} \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda}{\int_0^\infty \frac{\mathrm{d}B_\lambda}{\mathrm{d}T} \,\mathrm{d}\lambda}.$$

highlighting the fact that $1/\bar{\kappa}$ is a weighted average of $1/\kappa_\lambda$.

2. (a) Because

$$R = \lambda_n \xi_1,$$

it follows that

$$\lambda_n = \frac{1\,R_\odot}{6.897} = 1.01 \times 10^{10}\,\mathrm{cm}$$

(b) Using the equation

$$M = -4\pi\lambda_n^3\rho_c\xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1},$$

with $M = 1\,M_\odot$, it follows that

$$\rho_c = 76.3\,\mathrm{g\,cm^{-3}}.$$

(c) To find the central pressure, we use the polytropic relation:

$$P = K\rho^\gamma = K\rho^{1/n+1}$$

The constant K follows from the relation

$$\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{1/n-1}}{4\pi G}\right)\right]^{1/2};$$

rearranging,

$$K = \frac{4\pi G\lambda_n^2}{\rho_c^{1/n-1}(n+1)}$$

and so

$$K = 3.84 \times 10^{14}$$

(with funny units). Now using the polytropic relation,

$$P_c = 1.24 \times 10^{17}\,\mathrm{dyne\,cm^{-2}}.$$

(d) If the radiation pressure is negligible, the gas obeys the ideal equation of state,

$$P = \frac{\rho k T}{\mu m_H}$$

Solving for the central temperature,

$$T_c = 1.21 \times 10^7\,\mathrm{K}$$

3. From the various equations relating the mass and radius of polytropes, we find

$$R \propto \lambda_n$$

but

$$\lambda_n \propto K^{1/2}\rho_c^{1/2n-1/2}.$$

Due to the degenerate equation of state, $K = A$ is a constant — we don't have the liberty to choose it. Also, since $\gamma = 5/3$, it follows that $n = 1/(\gamma - 1) = 3/2$. Hence,

$$R \propto \rho_c^{-1/6}$$

But

$$M \propto \lambda_n^3\rho_c \propto R^3 R^{-6}$$

and so

$$R \propto M^{-1/3}.$$

4. See Fig. 1 for the plot. The gradient is estimated by eye as $\frac{d\log P}{d\log\rho} \equiv \gamma \approx 10/8$, from which we obtain $n = 1/(\gamma - 1) \approx 4$.

5. See Fig. 3 for the plot.

6. See Fig. 3 for the plot. The two luminosities differ in the core of the star ($r/R \lesssim 0.25$), with $L_{rad} < L_r$, because there a fraction of the total luminosity L_r is transported by convection rather than radiation.

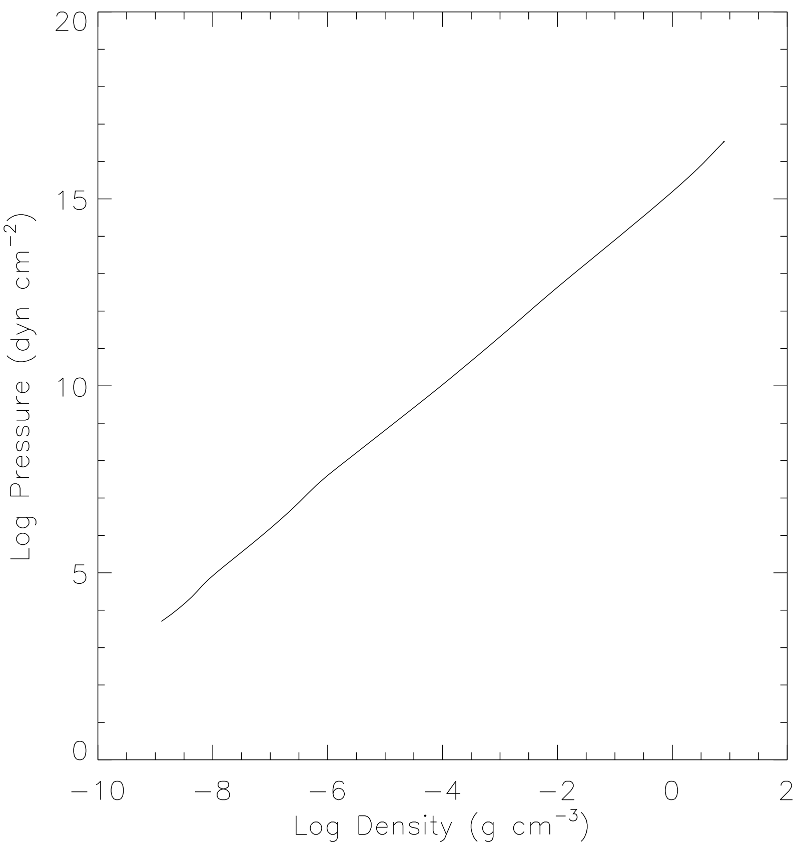

Figure 1: Pressure-density plot for the $10\,M_\odot$ model (Q4).

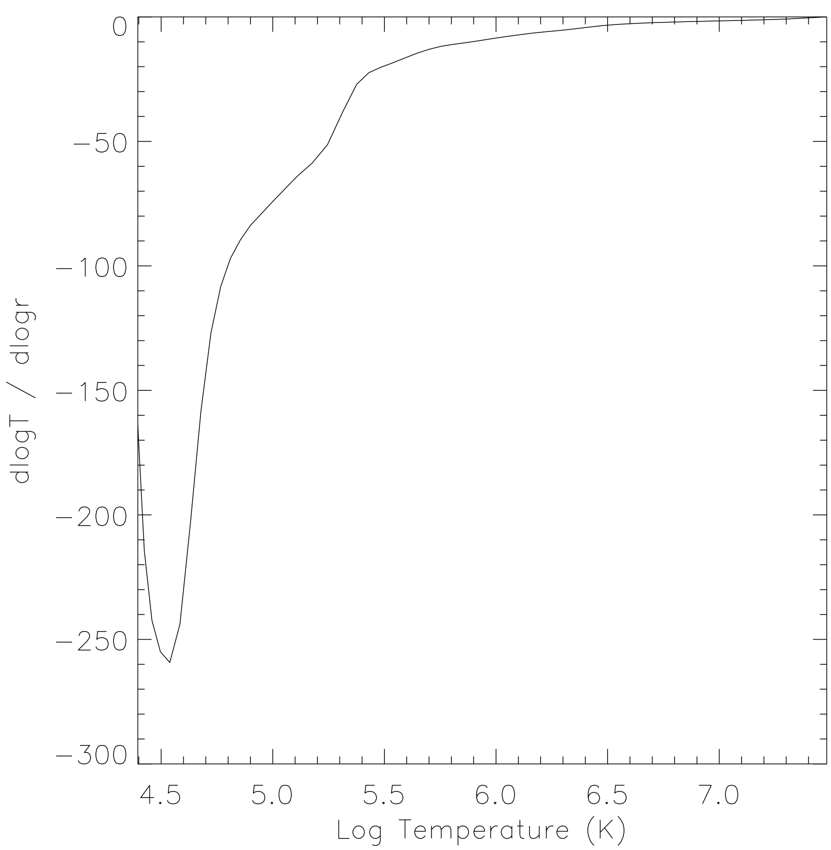

Figure 2: Temperature gradient plot for the $10\,M_{\odot}$ model (Q5).

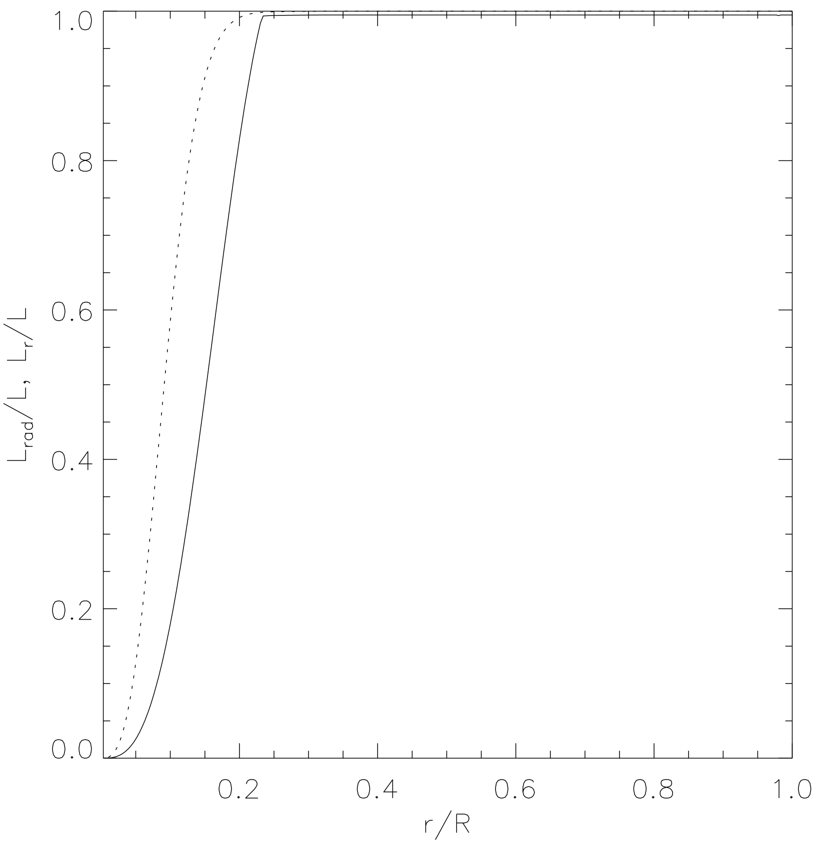

Figure 3: Luminosity plot for the $10\,M_\odot$ model (Q6). The solid line shows the radiative luminosity $L_{\rm rad}$, and the dotted line the total interior luminosity L_r.