Assignment 3 — due October 17$^{\text{th}}$ [*Revision* : 1.2]

1. For a gray atmosphere, suppose we approximate the directional (μ) dependence of the specific intensity using the first-order Taylor expansion,

$$I(\tau_{\rm v}, \mu) \approx I_0(\tau_{\rm v}) + I_1(\tau_{\rm v})\mu,$$

   where I_0 and I_1 depend on the vertical optical depth $\tau_{\rm v}$ but not on μ.

   (a) Derive expressions for the mean intensity $\langle I \rangle$, flux F and radiation pressure $P_{\rm rad}$, in terms of I_0 and I_1.

   (b) By comparing your expressions, show that the radiation field obeys the Eddington approximation $P_{\rm rad} = (4\pi/3c)\langle I \rangle$.

   (c) Within the Eddington approximation, the solution of the radiative transfer equation is

$$\langle I \rangle = \frac{3}{4\pi} F \left(\tau_{\rm v} + \frac{2}{3}\right)$$

   (see Notes 13). Use this solution to find an expression for I_0 as a function of I_1 and $\tau_{\rm v}$.

   (d) For $\tau_{\rm v} \gg 1$, demonstrate that $I_0 \gg I_1$. (This result justifies the use of the first-order Taylor expansion at large optical depths, showing that the radiation field becomes isotropic).

2. Consider a model atmosphere within the gray and Eddington approximations.

   (a) Assuming radiative equilibrium, write down an expression for the source function S as a function of vertical optical depth $\tau_{\rm v}$ and flux F.

   (b) Substitute this expression into the formal solution (see Notes 12), to calculate the *upward* specific intensity $I(\tau, \mu)$ as a function of F, $\tau_{\rm v}$ and direction $\mu > 0$.

   (c) Using your expression for the upward specific intensity, evaluate the upward component of the radiative flux,

$$F_+ = \int_0^{2\pi} \int_0^1 I\mu \,{\rm d}\mu \,{\rm d}\phi.$$

   (d) By using the relation

$$F = F_+ + F_-,$$

   find the corresponding downward component of the flux,

$$F_- = \int_0^{2\pi} \int_{-1}^0 I\mu \,{\rm d}\mu \,{\rm d}\phi.$$

   What is wrong with your result? What is the cause of this problem?

3. The Sun's corona is the extended, hot, low-density region above the Sun's photosphere, that is heated from below by the input of mechanical energy. The equation of hydrostatic equilibrium for the corona can be written as

$$\frac{{\rm d}P}{{\rm d}r} = -\frac{GM_\odot \rho}{r^2};$$

   this is different from the equation of hydrostatic equilibrium for a stellar atmosphere, in that (i) the vertical coordinate z has been replaced by the radial coordinate r; and (ii) the variation in the gravity g due to changes in r has been taken into account.

   (a) Assuming that the corona is an isothermal ideal gas with temperature T, solve the equation of hydrostatic equilibrium to find $P(r)$, expressing your result in terms of the surface pressure $P(R_\odot)$.

(b) Re-express your result in terms of the usual vertical coordinate $z = r - R_\odot$.

(c) Show that for $|z| \ll R_\odot$, the expression for $P(z)$ is equivalent to the one derived in class for an isothermal atmosphere.

(d) Show that for $z \to \infty$, the pressure asymptotes to a constant value.

(e) Calculate what this asymptotic value is, assuming $T \approx 10^6$ K; $P(R_\odot) \approx 1.4 \times 10^5\,\mathrm{dyne\,cm^{-2}}$, and $\mu \approx 10^{-24}$ g.

(f) How does this value compare to the pressure $P \approx 10^{-13}\,\mathrm{dyne\,cm^{-2}}$ in the Interstellar Medium? Can a hydrostatic corona exist — and if not, what instead happens?