Assignment 2 — Solutions [$Revision: 1.2$]

1. Because the enclosure is in thermal equilibrium, the radiation inside it will be blackbody radiation.

   (a) BB radiation is isotropic, and has a specific intensity given by the Planck function:

   $$I_\lambda = B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

   (b) Integrating over all wavelengths gives

   $$I = \int_0^\infty I_\lambda \mathrm{d}\lambda = \int_0^\infty B_\lambda \mathrm{d}\lambda = \frac{\sigma T^4}{\pi}$$

   where σ is the Stefan-Boltzmann constant (see O&C, eqn.3.28).

   (c) At the location of the hole, write the flux in terms of outward and inward intensities:

   $$F = 2\pi \int_{-1}^{1} I(\mu)\mu \mathrm{d}\mu = 2\pi \int_{-1}^{0} I_- \mu \mathrm{d}\mu + 2\pi \int_0^1 I_+ \mu \mathrm{d}\mu.$$

   The outward intensity I_+ is equal to the BB I given above, and the inward intensity I_- is equal to zero because the enclosure is in empty space. So:

   $$F = 2\pi \int_{-1}^{0} 0\mu \mathrm{d}\mu + 2\pi \int_0^1 \frac{\sigma T^4}{\pi} \mu \mathrm{d}\mu = \sigma T^4$$

   (d) This is the Stefan-Boltzmann equation.

[6 points]

2. Recall that in a visual binary system, we are able to see both stars, and watch them as they move about on the sky.

   (a) Because the circular orbit is tilted with respect to the line-of-sight, the apparent orbit will look like an an ellipse. Projection effects mean that the short (semi-minor) axis appears a factor of $\sin i$ smaller than the long (semi-major) axis (which is unaffected by the projection). So

   $$\beta_1 = \alpha_1 \sin i.$$

   Rearranging,

   $$\sin i = \frac{\beta_1}{\alpha_1}.$$

   (b) The radial velocity amplitude is the projection of the orbital speed onto the line-of-sight:

   $$v_{1\mathrm{r}} = v_1 \sin i,$$

   and so

   $$v_1 = \frac{v_{1\mathrm{r}}}{\sin i}.$$

   (c) The semi-major axis can be found by noting that, for a circular orbit, the orbital speed is given by

   $$v_1 = \frac{2\pi a_1}{P}$$

(i.e., distance travelled in one orbit divided by time taken). Rearranging,

$$a_1 = \frac{P v_1}{2\pi}.$$

Combining this with the expressions for v_1 and $\sin i$:

$$a_1 = \frac{P v_{1r}}{2\pi \sin i} = \frac{P v_{1r} \alpha_1}{2\pi \beta_1}$$

(d) The distance to the system is given by trigonometry:

$$d = \frac{a_1}{\alpha_1} = \frac{P v_{1r}}{2\pi \beta_1}$$

(e) The mass ratio of the system is given by ratio of semi-major axes (see notes):

$$\frac{m_1}{m_2} = \frac{a_2}{a_1}$$

Using the above expression for a_1, and a similar expression for a_2, gives

$$\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}} \frac{\alpha_2}{\alpha_1} \frac{\beta_1}{\beta_2}$$

The mass sum of the system is given by Kepler's third law (in its generalized form):

$$m_1 + m_2 = \frac{4\pi^2}{GP^2}(a_1 + a_2)^3.$$

Again, using the expressions for a_1 and a_2:

$$m_1 + m_2 = \frac{P}{2\pi G}\left(v_{1r}\frac{\alpha_1}{\beta_1} + v_{2r}\frac{\alpha_2}{\beta_2}\right)^3$$

[10 points]

3. A general overview of the (Harvard) spectral classification scheme is given in Table 8.1 of O&C; see also `http://ned.ipac.caltech.edu/level5/Gray/Gray_contents.html`.

   (a) Mid-type – strong Balmer lines are indicative of A-type stars
   (b) Insufficient information – weak Balmer lines are found in both early- and late-type stars
   (c) Late-type – TiO bands are seen in M-type stars
   (d) Early-type – He II is only seen in O-type stars
   (e) Late-type – the H & K lines are strongest in G- and K-type stars

[5 points]

4. This question makes extensive use of the expression relating the luminosity of a star to its radius and effective temperature:

$$L = 4\pi R^2 \sigma T_{\rm eff}^4$$

   (a) To plot the main sequence in the HRD, it is easiest first to write an expression for the main-sequence luminosity as a function of $T_{\rm eff}$. With

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^3$$

and

$$\frac{R}{R_\odot} \approx \frac{M}{M_\odot}$$

(valid for main-sequence stars), it follows that

$$\frac{R}{R_\odot} \approx \left(\frac{L}{L_\odot}\right)^{1/3}$$

Combining this with the luminosity-radius-temperature relation,

$$L = \left(4\pi \frac{R_\odot^2}{L_\odot^{2/3}} \sigma T_{\rm eff}^4\right)^3.$$

(b) Lines of constant stellar radius follow directly from the luminosity-radius-temperature relation above.

(c) Betelgeuse sits in the upper-right corner of the HRD; it is a red supergiant owing to its cool temperature (indicating red colors) and large radius ($R \approx 600\,R_\odot$).

(d) The lifetime is obtained by dividing the total nuclear energy released on the main sequence,

$$E = 0.7\% \times 0.1 \times Mc^2 = 7 \times 10^{-4} Mc^2$$

by the star's luminosity L. Using the mass-luminosity relation, this becomes

$$t = \frac{E}{L} = 7 \times 10^{-4} \frac{M_\odot c^2}{L_\odot^{2/7} L^{5/7}}.$$

(e) The main-sequence turnoff at $T_{\rm eff} = 10,000\,{\rm K}$ marks those stars in Praesepe that are reaching the end of their main-sequence lifetime. Reading from the plot, the age of these stars — and hence the age of the cluster — is $\log t/{\rm Myr} \approx 2.6$, or $t \approx 400\,{\rm Myr}$.

[10 points]

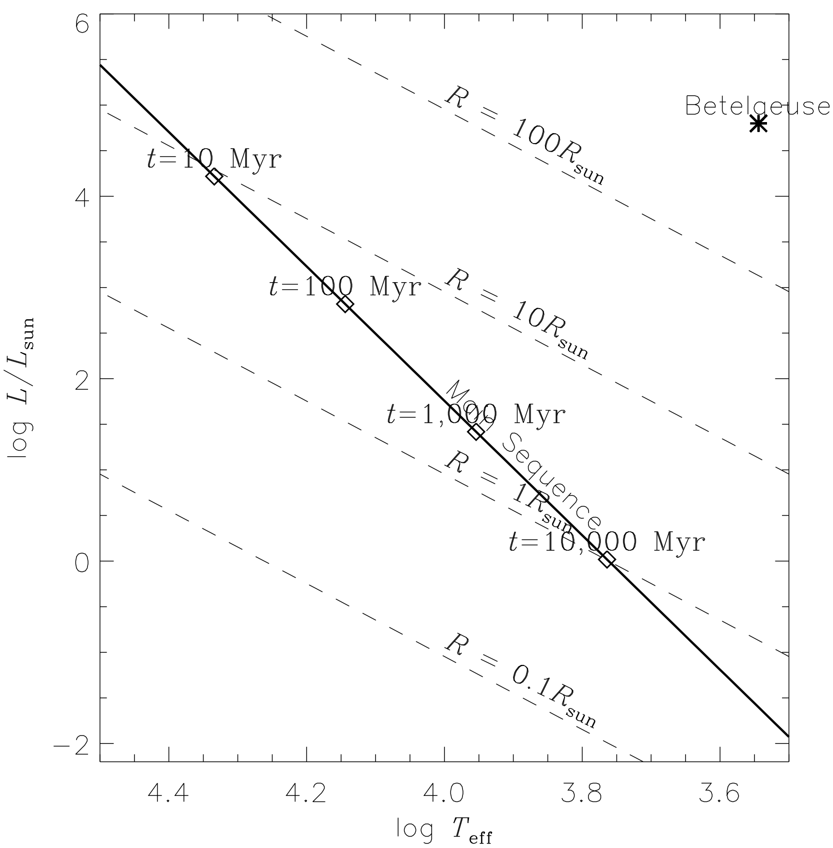

Figure 1: The HRD constructed in Question 4