Assignment 1 — Solutions [*Revision* : 1.2]

1. The Rayleigh-Jeans tail is the part of the blackbody spectrum where the photon energy $h\nu \equiv hc/\lambda$ is very small compared to the average energy per classical oscillator kT. The BB energy density formula

$$u_\lambda \mathrm{d}\lambda = \frac{8\pi hc/\lambda^5}{e^{hc/\lambda kT} - 1}\,\mathrm{d}\lambda$$

can be approximated in this limit by noting that the exponent $hc/\lambda kT$ becomes very small. For small x, the Taylor-series expansion for e^x gives

$$e^x = 1 + x + O(x^2).$$

Thus,

$$u_\lambda \mathrm{d}\lambda \approx \frac{8\pi hc/\lambda^5}{1 + hc/\lambda kT - 1}\,\mathrm{d}\lambda \approx \frac{8\pi kT}{\lambda^4}\,\mathrm{d}\lambda$$

which is the classical expression for the energy density in the R-J tail.

[3 points]

2.
   - At distance d from any source with luminosity L,

   $$F = \frac{L}{4\pi d^2}$$

   But from Stefan-Boltzmann law applied to Sun,

   $$L_\odot = 4\pi R_\odot^2 \sigma T_{\mathrm{eff}}^4$$

   So

   $$F = \frac{R_\odot^2}{d^2}\sigma T_{\mathrm{eff}}^4$$

   Plugging in numbers gives $F = 1.37 \times 10^6\,\mathrm{erg\,s^{-1}\,cm^{-2}}$.

   [2 points]

   - The total energy intercepted by the Earth every second is the solar flux times the Earth's cross-sectional area:

   $$\mathrm{d}E_{\mathrm{in}}/\mathrm{d}t = \pi R_\oplus^2 F.$$

   The total energy absorbed is then

   $$\mathrm{d}E_{\mathrm{ab}}/\mathrm{d}t = (1 - 0.3)\mathrm{d}E_{\mathrm{in}}/\mathrm{d}t = 0.7\pi R_\oplus^2 F.$$

   Plugging in the numbers gives $\mathrm{d}E_{\mathrm{ab}}/\mathrm{d}t = 1.22 \times 10^{24}\,\mathrm{erg\,s^{-1}}$.

   [2 points]

   - If Earth is a blackbody, its luminosity (energy radiated per second) will be given by

   $$L_\oplus = 4\pi R_\oplus^2 \sigma T_\oplus^4.$$

   The equilibrium temperature T_{eq} is the blackbody temperature at which the energy absorbed $\mathrm{d}E_{\mathrm{ab}}/dt$ is in balance with the energy radiated:

   $$0.7\pi R_\oplus^2 F = 4\pi R_\oplus^2 \sigma T_{\mathrm{eq}}^4,$$

   or

   $$T_{\mathrm{eq}} = \sqrt[4]{\frac{0.7F}{4\sigma}}$$

   Plugging in the numbers gives $T_{\mathrm{eq}} = 255\,\mathrm{K}$.

   [2 points]

- The greenhouse effect; the Earth's atmosphere behaves like a blanket, allowing through short-wavelength radiation from the Sun (which is peaked at around $5,000$ Å, but blocking the long-wavelength radiation re-emitted (which from Wien's law will have a peak at a wavelength $\approx 10\,\mu$m).

[**1 point**]

3. For the Johnson V (visible) passband, the apparent magnitude is defined by the equation

$$V = -2.5\log_{10}\int_0^\infty S_V(\lambda)F_\lambda\,\mathrm{d}\lambda + C_V$$

where $S_V(\lambda)$ is the V-band transmission function, F_λ is the monochromatic flux *at Earth*, and C_V is a constant we wish to find. Rearranging,

$$C_V = V + 2.5\log_{10}\int_0^\infty S_V(\lambda)F_\lambda\,\mathrm{d}\lambda.$$

Making the recommended approximation

$$\int_0^\infty S_X(\lambda)F_\lambda\mathrm{d}\lambda \approx \Delta\lambda F_{\lambda_0},$$

this becomes

$$C_V = V + 2.5\log_{10}\left(\Delta\lambda F_{\lambda_0}\right).$$

where λ_0 and $\Delta\lambda$ are the central wavelength and full width of the transmission function $S_V(\lambda)$.

[**3 points**]

To relate the observed monochromatic flux F_{λ_0} to the flux $F_{\lambda_0,*}$ at the surface of Vega, we use the inverse square law:

$$F_{\lambda_0} = \left(\frac{R}{d}\right)^2 F_{\lambda_0,*}$$

The monochromatic surface flux itself is calculated by assuming that Vega is a blackbody

$$F_{\lambda_0,*} = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda_0 kT_\mathrm{eff}}-1}$$

(**IMPORTANT NOTE**: there was an error in the definition of F_λ in the notes on blackbody radiation, involving a missing factor of 2π. As of revision 1.5 of the notes, this error is corrected; but since this correction post-dates the homework deadline, answers using the incorrect definition will still be given full points). Hence:

$$F_{\lambda_0} = \left(\frac{R}{d}\right)^2\frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda_0 kT_\mathrm{eff}}-1}$$

Plugging in the numbers gives a monochromatic V-band flux of $3.29\times10^{-9}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,Å^{-1}}$ (or $5.24\times10^{-10}\,\mathrm{erg\,s^{-1}\,cm^{-2}\,Å^{-1}}$ with the incorrect flux definition).

[**2 points**]

Combining this value with the bandwidth $\Delta\lambda = 890$ Å, and $V = 0.03$ for Vega, leads to the final result: $C_V = -13.8$. Another **IMPORTANT NOTE**: this result depends on the units we used to specify the integrated flux in the V band (in this, case, cgs). Had we used another system of units, the result would have been different. The table below gives the full results for the U, B and V bands, for calculations using both cgs and SI units (and with the correct/incorrect flux definitions).

	Correct F_λ		Incorrect F_λ	
	SI	cgs	SI	cgs
C_U	-20.9	-13.4	-22.9	-15.4
C_B	-20.8	-13.3	-22.8	-15.2
C_V	-21.3	-13.8	-23.3	-15.8

[3 points]

4. • Simbad gives $V = 2.005$.

[1 point]

• Using the given period/absolute magnitude formula, $M_V = -3.11$. Then, the absolute/apparent magnitude relation

$$V - M_V = 5\log_{10}\left(\frac{d}{10\,\text{pc}}\right)$$

gives the distance as $105\,\text{pc}$.

[2 points]

• The *Hipparcos* parallax leads to a distance

$$d = 1/0.00756'' = 132\,\text{pc}.$$

This is larger than the value found above, and also more accurate.

[2 points]

• Because of the way parallax is defined, the angular distance between the Earth and the Sun (as measured by the alien) will be the same as the parallax of Polaris (as measured from Earth). So, the angular separation is 7.56 milliarcsec.

[1 point]

• In this question, it's important that the same system of units as in question 3 is used (since the C_X values are units-dependent). The relevant equation for the V-band calculation (from previous question) is

$$V = -2.5\log_{10}\left(\Delta\lambda F_{\lambda_0}\right) + C_V,$$

and similarly for the other bands. As before,

$$F_{\lambda_0} = \left(\frac{R}{d}\right)^2 F_{\lambda_0,*}$$

and

$$F_{\lambda_0,*} = \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda_0 kT_{\text{eff}}} - 1}$$

Plugging in the relevant numbers (and using $288\,\text{K}$ for the Earth's effective temperature), we get the following results:

	U	B	V
Sun	11.4	10.8	10.4
Earth	163	138	114

[5 points]

• The V passband. The alien would do best to build an infrared telescope, because the contrast between the Earth and the Sun is least severe in the infrared.

[2 points]